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FROM INDEX SETS TO RANDOMNESS IN $\emptyset^n$: RANDOM REALS AND POSSIBLY INFINITE COMPUTATIONS PART II

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Abstract. We obtain a large class of significant examples of n-random reals (i.e., Martin-Löf random in oracle $\emptyset^{(n-1)}$) à la Chaitin. Any such real is defined as the probability that a universal monotone Turing machine performing possibly infinite computations on infinite (resp. finite large enough, resp. finite self-delimited) inputs produces an output in a given set $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$. In particular, we develop methods to transfer Σ^0_n or Π^0_n many-one completeness results of index sets to n-randomness of associated probabilities.

§1. Introduction. In Part I of this work [5] (Problem 1.8) we posed the following

QUESTION. For which sets $\mathscr{O}$ the probability that a universal monotone Turing machine U performing possibly infinite computations produces an output in $\mathscr{O}$ is n-random (i.e., Martin-Löf random in oracle $\emptyset^{(n-1)}$)?

The question can be considered either for infinite inputs with $U : \mathbf{2}^\omega \to \mathscr{X}$ a total map, or for finite inputs with $U : \mathbf{2}^{<\omega} \to \mathscr{X}$ a partial map with prefix-free domain (hereafter called self-delimited inputs). Following the idea developed in our paper [6], the question for finite inputs can also be considered for large enough finite inputs with $U : \mathbf{2}^{<\omega} \to \mathscr{X}$ a total map. This leads to three different probabilities denoted by $\Omega^\infty_U[\mathscr{O}]$ (infinite inputs), $\Omega^{\bowtie}_U[\mathscr{O}]$ (self-delimited finite inputs) and $\Omega^{\infty}_U[k, \mathscr{O}]$ (length $\geq k$ finite inputs).

Different notions of output can also be considered. The most natural choice is to consider outputs which are finite or infinite sequences of binary digits, i.e., $\mathscr{X} = \mathbf{2}^{\leq\omega} = \mathbf{2}^{<\omega} \cup \mathbf{2}^\omega$. As shown in [3], the range of $\Omega^\infty_U[\mathscr{O}]$, as a function of $\mathscr{O} \subseteq \mathbf{2}^{\leq\omega}$, is a finite union of closed intervals. In particular, $\Omega^\infty_U[\mathscr{O}]$ can be rational for non trivial simply defined $\mathscr{O}$'s. Nevertheless, in Part I [5], we proved 2-randomness of $\Omega^\infty_U[\mathscr{O}]$ and $\Omega^{\bowtie}_U[\mathscr{O}]$ under various conditions on $\mathscr{O}$.

In this paper, we consider outputs in $\mathfrak{P}(\mathbb{N})$, the family of all subsets of $\mathbb{N}$, a choice which allows significant transfer results from the theory of many-one degrees. A more general notion of output is possible, involving particular completions of computable partially ordered sets; it is developed in the forthcoming paper [7].

Main randomness results. Denoting by $\overline{\mathscr{O}}$ the complement of $\mathscr{O}$, Table 1 summarizes some randomness results given by Theorems 9.4, 9.7 and 9.11. The $\Omega^{\infty}_U[k, \mathscr{O}]$ column of Table 1 gives the type of randomness for both $\mathscr{O}$ and $\overline{\mathscr{O}}$. As for the $\Omega^\infty_U[\mathscr{O}]$ column of Table 1, the type of randomness of $\Omega^\infty_U[\overline{\mathscr{O}}]$ is the dual of that of $\Omega^\infty_U[\mathscr{O}]$ since $\Omega^\infty_U[\overline{\mathscr{O}}] = 1 - \Omega^\infty_U[\mathscr{O}]$. The $= 0$ and $= 1$ results of Table 1 are either

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	$\mathscr{O}$ **is the set of** X**'s such that**	$\Omega_U^{\propto}[k,\mathscr{O}]$ / $\Omega_U^{\propto}[k,\overline{\mathscr{O}}]$	$\Omega_U^{\infty}[\mathscr{O}]$	$\Omega_U^{\bowtie}[\mathscr{O}]$ / $\Omega_U^{\bowtie}[\overline{\mathscr{O}}]$
0	all X's	$=1/=0$	$=1$	$\Sigma_2^0/=0$
1a	$X \supseteq A$ A finite, $A \neq \emptyset$	Σ_1^0/Σ_2^0	Σ_1^0	Σ_2^0/Σ_2^0
1b	$X \supseteq A$ A infinite comput. or Σ_2^0	Σ_3^0/Σ_2^0	Π_2^0	$?/\Sigma_2^0$
2a	$X \subseteq A$ $A \neq \mathbb{N}$ comput. or Π_1^0	Σ_2^0/Σ_1^0	Π_1^0	Σ_2^0/Σ_2^0
2b	$X \subseteq A$ A Π_2^0 and not Π_1^0	$?/?$	$?$	$?/\Sigma_2^0$
3	$X \subseteq Y$	Σ_3^0/Σ_2^0	Π_2^0	$?/\Sigma_2^0$
4a	$X = \emptyset$	Σ_2^0/Σ_1^0	Π_1^0	Σ_2^0/Σ_2^0
4b	$X = A$ $A \neq \emptyset$ finite	$?/?$	$?$	Σ_2^0/Σ_2^0
4c	$X = A$ A infinite c.e.	Σ_3^0/Σ_2^0	$?$	$?/\Sigma_2^0$
4d	$X = A$ A not c.e.	$=0/=1$	$=0$	$=0/=1$
5	$X = Y$	Σ_3^0/Σ_2^0	Π_2^0	$?/\Sigma_2^0$
6a	X is finite	Σ_2^0/Σ_3^0	Σ_2^0	$\Sigma_2^0/?$
6b	$\lvert X\rvert \leq p$ $p \in \mathbb{N}$	Σ_2^0/Σ_1^0	Π_1^0	Σ_2^0/Σ_2^0
6c	$\forall p\ \lvert X \cap \{0,..,p\}\rvert \leq \lvert Y \cap \{0,..,p\}\rvert$	Σ_3^0/Σ_2^0	Π_2^0	$?/\Sigma_2^0$
7	$X \subseteq \mathbb{N}^2$ is a linear ordering	Σ_3^0/Σ_2^0	Π_2^0	$?/\Sigma_2^0$
8	X is cofinite	Σ_3^0/Σ_4^0	Σ_3^0	$?/?$
9a	X is c.e.	$=1/=0$	Σ_3^0	$\Sigma_2^0/=0$
9b	X is boolean combination of c.e.	$=1/=0$	Σ_3^0	$\Sigma_2^0/=0$
9c	Idem with X A-c.e. if A low c.e.	$=1/=0$	Σ_3^0	$\Sigma_2^0/=0$
10a	X is computable	Σ_3^0/Σ_4^0	Σ_3^0	$?$
10b	the complement of X is c.e.	Σ_3^0/Σ_4^0	Σ_3^0	$?$
11a	X is simple	Σ_4^0/Σ_3^0	$?$	$?$
11b	X is c.e. not simple	Σ_3^0/Σ_4^0	Σ_3^0	$?$
12	X is maximal	Σ_5^0/Σ_4^0	$?$	$?$
13	X is atomless	Σ_6^0/Σ_5^0	$?$	$?$
14a	$A \leq_{Turing} X$ where A is low c.e.	Σ_3^0/Σ_4^0	Σ_3^0	$?$
14b	$A \equiv_{Turing} X$ where A is low c.e.	Σ_3^0/Σ_4^0	$?$	$?$
14c	Idem with X c.e.	Σ_3^0/Σ_4^0	$?$	$?$
15a	$A \leq_{Turing} X$ where A is high c.e.	Σ_4^0/Σ_5^0	Σ_4^0	$?$
15b	$A \equiv_{Turing} X$ where A is high c.e.	Σ_4^0/Σ_5^0	$?$	$?$
15c	Idem with X c.e.	Σ_4^0/Σ_5^0	$?$	$?$
16a	$X \leq_{Turing} Y$	Σ_4^0/Σ_5^0	Σ_4^0	$?$
16b	$X \equiv_{Turing} Y$	Σ_4^0/Σ_5^0	$?$	$?$
17a	X is c.e. and n-low	$\Sigma_{n+3}^0/\Sigma_{n+4}^0$	$?$	$?$
17b	X is c.e. and n-high	$\Sigma_{n+4}^0/\Sigma_{n+5}^0$	$?$	$?$
18	$X \in Set(w)$ with $w \in \{\text{FIN},\text{COF}\}^{<\omega}$, $\lvert w\rvert_{FIN} = i$, $\lvert w\rvert_{COF} = j$ (cf. §4.2)	Σ_{i+2j+1}^0 / Σ_{i+2j+2}^0	Σ_{i+2j+1}^0	$?$
19	$X \subseteq \mathbb{N}^2$ is well-founded	$\Pi_1^1/?$	Π_1^1	$?$

Σ_n^0 (resp. Π_n^0) in the last three columns means that $\Omega_U^{\cdot\cdot}[\mathscr{O}]$ is n-random with Σ_n^0 (resp. Π_n^0) left cut in the set of rational numbers.

TABLE 1. Some randomness results implied by Theorems 9.4, 9.7, 9.11.

trivial (line 0) or straightforward consequences of Propositions 2.3 and 2.5. Question marks in Table 1 are open questions for which the methods of this paper do not apply or which are technically challenging. Observe that we have few randomness results for families $\mathscr{O}$ which are $\Sigma^0_n \wedge \Pi^0_n$ (in the sense of the Scott arithmetical hierarchy on $\mathfrak{P}(\mathbb{N})$ introduced in §3.1): one with $n = 1$ for $\Omega_U^{\bowtie}[\mathscr{O}]$ (line 4b) and one with $n = 3$ for $\Omega_U^{\propto}[k,\mathscr{O}]$ (line 11a).

The material is organized as follows. §2 formalizes three notions of computable maps associated to possibly infinite computations with outputs in $\mathfrak{P}(\mathbb{N})$ and inputs in $\mathbf{2}^\omega$ or in $\mathbf{2}^{<\omega}$ (all words) or in a prefix-free set of words. §3 introduces the pertinent topology on $\mathfrak{P}(\mathbb{N})$, which is a non Hausdorff (but still T_0) weakening of the Cantor topology. The associated Scott arithmetical hierarchy, i.e., the effectivization of the finite levels of the Scott Borel hierarchy, does not coincide with that associated to the Cantor topology. Definability with respect to that hierarchy is studied in §4. In particular, we give a general theorem to get syntactical complexity of "index like" definable subsets of $\mathfrak{P}(\mathbb{N})$ (cf. Corollary 4.5).

§5 is devoted to three concepts of hard subsets $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ associated to the three types of computable maps. Such hardness is relative to subsets of the Cantor space $\mathbf{2}^\omega$ or of the discrete space $\mathbf{2}^{<\omega}$. In the case of $\mathbf{2}^\omega$, the so-called effective Wadge hardness is the effective analog of the classical Wadge hardness in descriptive set theory (cf. Wadge [19]).

These concepts of hardness for $\Sigma^0_n(\mathbf{2}^\omega)$ or for $\Sigma^0_n(\mathbf{2}^{<\omega})$ are key hypotheses in the basic randomness Theorems 9.2, 9.5 and 9.8, asserting n-randomness of the reals $\Omega_U^{\infty}[\mathscr{O}]$, $\Omega_U^{\propto}[k,\mathscr{O}]$ and $\Omega_U^{\bowtie}[\mathscr{O}]$ (associated to the three types of computable maps) provided $\mathscr{O}$ is Scott Σ^0_n definable. These theorems are direct generalizations of Theorem 1.11 of Part I [5] where we proved 2-randomness. The core of their proofs brings nothing new, being uniform for every $n \geq 1$, only the proof for $\Omega_U^{\propto}[k,\mathscr{O}]$ needs an extra argument similar to that developed in our paper [6]. Equality $\Omega_U^{\infty}[\overline{\mathscr{O}}] = 1 - \Omega_U^{\infty}[\mathscr{O}]$ allows for randomness results with Scott Π^0_n families $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$.

The case of $\Omega_U^{\bowtie}[\mathscr{O}]$ is very particular: the associated (so-called) open special hardness is rather trivial for Σ^0_2, impossible for Σ^0_n, $n \geq 3$, and the weaker measure special hardness is problematic (cf. § 8.3). This is why we have no application of the a priori useful equality $\Omega_U^{\bowtie}[\overline{\mathscr{O}}] = \Omega_U^{\bowtie}[\mathfrak{P}(\mathbb{N})] - \Omega_U^{\bowtie}[\mathscr{O}]$.

Although there is no such equality with $\Omega_U^{\propto}[k,\mathscr{O}]$, a much better result holds in that case since then open second order many-one Π^0_n hardness implies Σ^0_{n+1} hardness (cf. Theorem 5.11). Whence the surprising $\Sigma^0_n/\Sigma^0_{n+1}$ and $\Sigma^0_{n+1}/\Sigma^0_n$ pairs in the $\Omega_U^{\propto}[k,\mathscr{O}]$ column of Table 1.

Sections §6, §7and §8 prove sufficient conditions (which are also necessary in a few cases) for each of the three hardness notions for families of subsets of $\mathfrak{P}(\mathbb{N})$. At level 1, second order many-one hardness and effective Wadge hardness of Scott open families of $\mathfrak{P}(\mathbb{N})$ are both equivalent to the condition $\mathscr{O} \neq \emptyset \wedge \emptyset \notin \mathscr{O}$ (cf. Propositions 6.1, 7.1). At level 2, effective Wadge hardness is characterized (cf. Proposition 7.3) and implies second order many-one hardness (cf. Proposition 6.5). A sufficient very efficient condition is the "chain property" (cf. Proposition 6.4). As for open special hardness, level 1 and 2 are both equivalent to the condition that $\mathscr{O}$ contains a c.e. set (cf. Proposition 8.1). These results lead to significant 1-random and 2-random reals $\Omega_U^{\propto}[k,\mathscr{O}]$ and $\Omega_U^{\infty}[\mathscr{O}]$ and 2-random Σ^0_2 reals $\Omega_U^{\bowtie}[\mathscr{O}]$ (cf. Table 1).

Most important, one can prove (cf. Theorem 6.9) that the usual many-one hardness of the index set of a family $\mathscr{O}$ of c.e. sets is equivalent to second order many-one hardness of $\mathscr{O}$. This gives a very powerful tool to get significant n-random Σ^0_n reals $\Omega^\infty_U[k,\mathscr{O}]$ (cf. Table 1).

A version of such a general transfer theorem from many-one hardness of index sets also holds for effective Wadge hardness (cf. Theorem 7.20). It takes advantage of the fact that most of the many-one hardness results for index sets of c.e. sets relativize –with trivial changes in the proof– to index sets of A-c.e. sets uniformly in the oracle $A \subseteq \mathbb{N}$. This is the source of randomness of $\Omega^\infty_U[\mathscr{O}]$ in lines 7,8, 18, 19 of Table 1. However, this transfer theorem is more suited for families defined in a set theoretic way. For families $\mathscr{O}$ defined by computability conditions, effective à la Wadge hardness happens to be the most difficult to prove at levels ≥ 3. We show how to transfer Rogers and Yates classical proofs of many-one hardness of index sets at levels 3 and 4 into effective Wadge hardness of associated subsets of $\mathfrak{P}(\mathbb{N})$, cf. Theorems 7.6, 7.18. This leads to significant n-random reals $\Omega^\infty_U[\mathscr{O}]$, for $n = 3, 4$ (cf. Table 1, lines 9abc,10ab,14a,15a,16a).

NOTATION 1.1. The alphabet $\{0,1\}$ is denoted with $\mathbf{2}$ and, as usual, we write $\mathbf{2}^{\leq n}, \mathbf{2}^{\geq n}$ and $\mathbf{2}^{<\omega}$ to denote, respectively, the set of all words up to length n, the set of all words with length at least n and that of all finite words. The length of a word w is denoted by $|w|$ and the cardinal of a finite set X is denoted by $|X|$. $\mathfrak{P}(X)$ denotes the power set of X and $\mathfrak{P}_{<\omega}(X)$ is the set of all finite subsets of X whereas $\mathbf{2}^\omega$ denotes the set of all infinite words, i.e, the Cantor space. The Lebesgue measure of a subset $\mathscr{X}$ of the Cantor space $\mathbf{2}^\omega$ is denoted by $\mu(\mathscr{X})$.

Let $A \subseteq \mathbb{N}$ be some fixed oracle. We denote by $(W_e)_{e\in\mathbb{N}}$ and $(W^A_e)_{e\in\mathbb{N}}$ standard enumerations of computably enumerable sets (in short c.e.) and A-c.e. subsets of $\mathbb{N}$ or of $\mathbb{N}^d$, (the dimension d being clear from context).

§2. Three notions of computable maps into $\mathfrak{P}(\mathbb{N})$ for infinite computations.

2.1. Monotone machines with outputs in $\mathfrak{P}(\mathbb{N})$ or $\mathfrak{P}(\mathbb{N}^d)^\ell$. We consider Turing machines with finite or infinite binary words as inputs which enumerate sets of natural numbers. These are monotone Turing machines performing possibly infinite computations which output integers from time to time. The input tape is one-way read-only and the output tape –which receives integers, not digits (whatever be the coding of these integers)– is one-way write-only (i.e., no erasing nor over-writing is possible). We care neither about the order of appearance of these integers nor the number of times that a given integer appears. Thus, the resulting output of such a possibly infinite computation is a finite or infinite set of integers.

Considering monotone machines with ℓ output tapes that receive d-tuples of integers, we similarly get outputs in $\mathfrak{P}(\mathbb{N}^d)^\ell$. This straightforward extension is needed for results in lines 3, 5, 6c, 7, 16b, 18, 19 of Table 1.

Throughout the paper the term *monotone Turing machine* means such machines. To simplify notations, we shall consider the case $d = \ell = 1$. All notions and results stated for $\mathfrak{P}(\mathbb{N})$ go through in an obvious way in the general case of $\mathfrak{P}(\mathbb{N}^d)^\ell$ (for Theorem 2.8 and Proposition 2.9, we give proofs for the general case).

2.2. Three notions of computable maps. First, we consider the simplest definition of possibly infinite computations on finite or infinite inputs.

DEFINITION 2.1 (Computable total maps). A total map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ (resp. $F : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$) is computable if there exists some monotone Turing machine M such that, for every $u \in \mathbf{2}^{<\omega}$ (resp. $\alpha \in \mathbf{2}^{\omega}$), $F(u)$ (resp. $F(\alpha)$) is the subset of $\mathbb{N}$ output by M through a possibly infinite computation on input u (resp. α).

If finite inputs are restricted to a prefix-free set we obtain the following.

DEFINITION 2.2 (Self-delimited partial computable maps). Let $k, \ell \geq 1$. A partial map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ is self-delimited partial computable if there exists some monotone Turing machine M such that, for every $u \in \mathbf{2}^{<\omega}$,

i. $F(u)$ is defined if and only if the input head reads u entirely during the computation and does not visit any cell beyond u.
ii. When defined, $F(u)$ is the subset of $\mathbb{N}$ output by M through a possibly infinite computation on input u.

The following results are straightforward.

PROPOSITION 2.3. Let $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ be either a computable total map or a self-delimited partial computable map. The range of F is included in the family of computably enumerable (c.e.) subsets of $\mathbb{N}$. In particular, $F^{-1}(\mathscr{O})$ is empty if $\mathscr{O}$ contains no c.e. set.

PROPOSITION 2.4. The domain of a self-delimited partial computable map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ is a prefix-free subset of $\mathbf{2}^{<\omega}$ which is $\Sigma^0_1 \wedge \Pi^0_1$ in $\mathbf{2}^{<\omega}$.

The version of Proposition 2.3 for maps from $\mathbf{2}^{\omega}$ is as follows.

PROPOSITION 2.5. Let $F : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$ be a computable total map. For any $\alpha \in \mathbf{2}^{\omega}$ the image $F(\alpha)$ is α-c.e. In particular, $F^{-1}(\{A\})$ has Lebesgue measure zero if A is not a c.e. set.

PROOF. The first assertion is obvious. For the second one, apply Sacks' result which insures that $\{\alpha \in \mathbf{2}^{\omega} \mid A \text{ is } \alpha\text{-c.e.}\}$ has measure 0 if A is not c.e. (Sacks [14] p.156 Theorem 2). ⊣

2.3. Universality by adjunction. An enumeration theorem holds for all the above types of computable maps. We state it in terms of *universality by adjunction*, a notion heavily used in Part I [5].

NOTATION 2.6. If $F : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$ and $i \in \mathbb{N}$, we denote by $F_i : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$ the map such that $F_i(\alpha) = F(0^i 1\alpha)$ for all α.

If $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ and $i \in \mathbb{N}$, we denote by $F_i : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ the map such that $F_i(u) = F(0^i 1u)$ for all $u \in \mathbf{2}^{<\omega}$. Observe that if F has prefix-free domain then so have all F_i's.

DEFINITION 2.7 (Universality by adjunction).

1. A computable total map $U_{\propto} : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ (resp. $U_{\infty} : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$) is universal by adjunction if for any computable total map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ (resp. $F : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$) there exists $i \in \mathbb{N}$ such that $F = (U_{\propto})_i$ (resp. $F = (U_{\infty})_i$).
2. A self-delimited computable partial map $U_{\bowtie} : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ is universal by adjunction if for any self-delimited partial computable map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ there exists $i \in \mathbb{N}$ such that $F = (U_{\bowtie})_i$.

THEOREM 2.8. 1. There exists computable total maps $U_\propto : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ and $U_\infty : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ which are universal by adjunction.

2. There exists a self-delimited partial computable map $U_{\bowtie} : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ which is universal by adjunction. Moreover, the domain of $U_{\bowtie}$ can be taken to be included in $0^{<\omega}1\mathbf{2}^{<\omega}$.

PROOF. We give the proof in the general case $\mathfrak{P}(\mathbb{N}^d)^\ell$ instead of $\mathfrak{P}(\mathbb{N})$ since it needs a simple additional trick.

Case of total maps. Let $(M_i)_{i\in\mathbb{N}}$ be a computable enumeration of monotone Turing machines with ℓ output tapes and let $F_i : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N}^d)^\ell$ (resp. $F_i : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N}^d)^\ell$) be the computable map associated to M_i. Set $U_\propto(0^i) = (\emptyset, ..., \emptyset)$ and $U_\propto(0^i1u) = F_i(u)$. (resp. $U_\infty(0^\omega) = (\emptyset, ..., \emptyset)$ and $U_\infty(0^i1\alpha) = F_i(\alpha)$). It is straightforward to see that $U_\propto$ (resp. U_∞) is computable.

Case of self-delimited maps. Define $U_{\bowtie}$ with domain $\{0^i1u \mid u \in dom(F_i)\}$ so that $U_{\bowtie}(0^i1u) = F_i(u)$. ⊣

PROPOSITION 2.9. 1. If $U_\infty : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ is total computable universal by adjunction then U_∞ is surjective.

2. If $U_\propto : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ (resp. $U_{\bowtie} : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$) is total computable (resp. self-delimited partial computable) universal by adjunction then its range is the family of c.e. subsets of $\mathbb{N}$.

PROOF. Again, we give the proof in the general case $\mathfrak{P}(\mathbb{N}^d)^\ell$ instead of $\mathfrak{P}(\mathbb{N})$.

Case $U_\infty : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N}^d)^\ell$. Since U_∞ is universal, its range contains the range of any computable total map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N}^d)^\ell$. Observe that there exists a surjective such F, for instance $F(\alpha) = (\{f(i) \mid \alpha(i\ell + j) = 1\})_{j=0,...,\ell-1}$ where f is some fixed computable bijection $\mathbb{N} \to \mathbb{N}^d$.

Case $U_{\bowtie} : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N}^d)^\ell$ *(resp.* $U_\propto : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N}^d)^\ell$*).* Same proof: consider the total (resp. partial) computable map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N}^d)^\ell$ such that $F(0^{i_1}1...10^{i_\ell}1v) = (W_{i_1}, ..., W_{i_\ell})$ and $F(w) = (\emptyset, ..., \emptyset)$ (resp. F undefined) if w has less than ℓ 1's. ⊣

§3. Topology and arithmetical hierarchy on $\mathfrak{P}(\mathbb{N})$ and $\mathfrak{P}(\mathbb{N}^d)^\ell$.

3.1. Scott topology and arithmetical hierarchy. The pertinent topology on $\mathfrak{P}(\mathbb{N}^d)$ with respect to computable maps $\mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N}^d)$ is the Scott topology (Scott, [15] 1972). In order to get a proper hierarchy, the usual definition of the finite levels of the Borel hierarchy has to be distorted at level 2, cf. Selivanov [16], 2005.

DEFINITION 3.1. 1. (*Scott topology on* $\mathfrak{P}(\mathbb{N})$) If $A \subset \mathbb{N}$, let $\mathscr{B}_A = \{X \mid X \supseteq A\}$. Let $\mathscr{B}_{Scott} = \{\mathscr{B}_A \mid A \subset \mathbb{N} \text{ is finite}\}$. The open sets of the Scott topology on $\mathfrak{P}(\mathbb{N})$ are all arbitrary unions of sets in the basis $\mathscr{B}_{Scott}$.

2. The *Scott arithmetical hierarchy on the* $\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m$'s is defined by induction on $n \in \mathbb{N}$ as follows: let $\mathscr{X} \subset \mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m$ and $n \in \mathbb{N}$,

$$\mathscr{X} \in \Sigma^0_1(\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m) \Leftrightarrow \mathscr{X} = \bigcup_{(A,\vec{i})\in\mathscr{C}} \mathscr{B}_A \times \{\vec{i}\}$$
$$\text{where } \mathscr{C} \text{ is a c.e. subset of } \mathfrak{P}_{<\omega}(\mathbb{N}) \times \mathbb{N}^m,$$
$$\mathscr{X} \in \Sigma^0_2(\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m) \Leftrightarrow \mathscr{X} = \{(X, \vec{i}) \mid \exists j\ (X, \vec{i}, j) \in (\mathscr{Y} \setminus \mathscr{Z})\}$$
$$\text{where } \mathscr{Y}, \mathscr{Z} \in \Sigma^0_1(\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^{m+1}),$$

$$\mathscr{X} \in \Sigma^0_{n+3}(\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m) \Leftrightarrow \mathscr{X} = \{(X, \vec{i}) \mid \exists j\ (X, \vec{i}, j) \in \mathscr{Y}\}$$
$$\text{where } \mathscr{Y} \in \Pi^0_{n+2}(\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^{m+1}),$$
$$\mathscr{X} \in \Pi^0_n(\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m) \Leftrightarrow (\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m) \setminus \mathscr{X} \in \Sigma^0_n(\mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m).$$

We shall also say that $\mathscr{X} \subseteq \mathfrak{P}(\mathbb{N})$ is Scott Σ^0_n or Π^0_n when it is in $\Sigma^0_n(\mathfrak{P}(\mathbb{N}))$ or in $\Pi^0_n(\mathfrak{P}(\mathbb{N}))$.

3. The Scott topology on $\mathfrak{P}(\mathbb{N})^\ell$ is the ℓ-th power of the Scott topology on $\mathfrak{P}(\mathbb{N})$. The Scott arithmetical hierarchy on $\mathfrak{P}(\mathbb{N})^\ell$ is obtained similarly as above. Replacing $\mathfrak{P}(\mathbb{N})^\ell$ by $\mathfrak{P}(\mathbb{N}^d)^\ell$ is straightforward.

Remark 3.2. The Scott topology on $\mathfrak{P}(\mathbb{N})$ is not Hausdorff. However, it is T_0, i.e., for any pair of distinct elements $X, Y \in \mathfrak{P}(\mathbb{N})$, there is an open set which contains X and not Y or there is an open set which contains Y and not X. In fact, suppose $X \neq Y$ and let i be in the symmetric difference of X and Y. Then $\mathscr{B}_{\{i\}}$ contains either X or Y but not both.

Remark 3.3. As noticed by Selivanov [16], the Scott arithmetical hierarchy on $\mathfrak{P}(\mathbb{N})$ does not coincide with the arithmetical hierarchy on the Cantor space $\mathbf{2}^\omega$ (modulo the standard identification of $\mathfrak{P}(\mathbb{N})$ with $\mathbf{2}^\omega$). In fact, for all n,

$$\Sigma^0_n(\mathfrak{P}(\mathbb{N})) \subsetneq \Sigma^0_n(\mathbf{2}^\omega) \subsetneq \Sigma^0_{n+1}(\mathfrak{P}(\mathbb{N})).$$

The same is true with the Borel hierarchy. For instance, $\mathscr{X} = \mathfrak{P}(\mathbb{N}) \setminus \{\mathbb{N}\}$, defined by the formula $\exists x\ x \notin X$, is $\Sigma^0_1(\mathbf{2}^\omega)$ and Scott Σ^0_2 but neither Scott open nor Scott closed. However, the infinite levels of the Borel hierarchies and the projective hierarchies on $\mathfrak{P}(\mathbb{N})$ and $\mathbf{2}^\omega$ coincide.

A straightforward induction shows that one can compute the Scott level from a defining first-order formula (with first-order and second order variables), in the usual way, provided that $x \in X$ be considered as Scott Σ^0_1.

Proposition 3.4. Let $R_i(x_1, \dots, x_p)$, $i \in \mathbb{N}$ be predicates to be interpreted by computable relations on $\mathbb{N}$. Let $\phi(X_1, \dots, X_\ell, x_1, \dots, x_k)$ be a first-order formula built on the atomic predicates $x \in X$ plus the R_i's. Let $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})^\ell \times \mathbb{N}^k$ be the family defined by ϕ. Let $\psi(x_1, \dots, x_p)$ be the formula obtained from $\phi(X_1, \dots, X_\ell, x_1, \dots, x_k)$ by substituting to any atomic formula $x \in X$ the existential formula $\exists t\ P(x, X, t)$ where P is a new atomic predicate (intuitive meaning: x is put in X at time t). If the syntactical complexity of ψ is Σ_n (resp. Π_n) then $\mathscr{O}$ is Scott Σ^0_n (resp. Scott Π^0_n).

3.2. Computability, continuity and arithmetical hierarchy. The following results partly transfer to $\mathfrak{P}(\mathbb{N})$ and $\mathfrak{P}(\mathbb{N}^d)^\ell$ what was proved for $\mathbf{2}^{\leq\omega}$ in [4], Theorem 81.

Theorem 3.5. Every computable total map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ is continuous relative to the Cantor topology on $\mathbf{2}^\omega$ and the Scott topology on $\mathfrak{P}(\mathbb{N})$. Idem with $\mathfrak{P}(\mathbb{N}^d)^\ell$.

Proof. Let M be a monotone machine which computes F. Let $A \subset \mathbb{N}$ be finite. If $F(\alpha) \in \mathscr{B}_A$ then there exists some step t such that, on input α, M has already output all elements of A. Since at step t the input head has read at most the t first symbols of α, we see that if $\beta \upharpoonright t = \alpha \upharpoonright t$ then $F(\beta) \supseteq A$. Thus, $(\alpha \upharpoonright t)\mathbf{2}^\omega$ is a neighborhood of α mapped into $\mathscr{B}_A$ by F. This proves continuity of F. $\dashv$

	$\mathscr{O}$ **is the set of** X**'s such that**	$\mathscr{O}$ **is**	**conditions**
1a	$X \supseteq A$	Σ^0_1	A finite, $A \neq \emptyset$
1b	$X \supseteq A$	Π^0_2	A infinite comput. or Σ^0_2
2a	$X \subseteq A$	Π^0_1	$A \neq \mathbb{N}$ comput. or Π^0_1
2b	$X \subseteq A$	Π^0_2	A Π^0_2 and not Π^0_1
3	$X \subseteq Y$	Π^0_2	
4a	$X = \emptyset$	Π^0_1	
4b	$X = A$	$\Sigma^0_1 \wedge \Pi^0_1$	A finite, $A \neq \emptyset$
4c	$X = A$	Π^0_2	A infinite comput. or Σ^0_1
4d	$X = A$	Π^0_2	A infinite Δ^0_2 not Σ^0_1
5	$X = Y$	Π^0_2	
6a	X is finite	Σ^0_2	
6b	$\lvert X\rvert \leq p$	Π^0_1	$p \in \mathbb{N}$
6c	$\forall p\ \lvert X \cap \{0, .., p\}\rvert \leq \lvert Y \cap \{0, .., p\}\rvert$	Π^0_2	
7	$X \subseteq \mathbb{N}^2$ is a linear ordering	Π^0_2	
8	X is cofinite	Σ^0_3	
18	$X \in \mathit{Set}(w)$	Σ^0_{i+2j+1}	$w \in \{\mathit{FIN}, \mathit{COF}\}^{<\omega}$ (Def.4.2) $\lvert w\rvert_{\mathit{FIN}} = i, \quad \lvert w\rvert_{\mathit{COF}} = j$
19	$X \subseteq \mathbb{N}^2$ is well-founded	Π^1_1	

TABLE 2. Complexity of set theoretic families $\mathscr{O}$ in the Scott arithmetical hierarchy.

THEOREM 3.6. Let $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ be Scott Σ^0_n.

1. If $n \geq 1$ and $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ is a computable total map then $F^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^\omega)$.
2. If $n \geq 1$ and $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ is a computable total map then $F^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^{<\omega})$.
3. If $n \geq 2$ and $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ is a self-delimited partial computable map then $F^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^{<\omega})$.
4. Idem with Π^0_n or Π^1_1 in place of Σ^0_n and $\mathfrak{P}(\mathbb{N}^d)^\ell$ in place of $\mathfrak{P}(\mathbb{N})$.

PROOF. 1. Effectivization of the argument of the proof of Theorem 3.5 yields the Σ^0_1 case and the extension to sequences of sets: if $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N}) \times \mathbb{N}^m$ is Scott Σ^0_1 then $\{(\alpha, \vec{i}) \mid (F(\alpha), \vec{i}) \in \mathscr{O}\} \in \Sigma^0_1(\mathbf{2}^\omega \times \mathbb{N}^m)$. Commutation of F^{-1} with set difference and countable unions reduces all cases to the Σ^0_1 case.

2. Argue as in the proof of Theorem 3.5 with input $u \in \mathbf{2}^{<\omega}$ in place of $\alpha \in \mathbf{2}^\omega$. Then argue as in point 1.

3. Again argue as in the above proof of Theorem 3.5 with input $u \in \mathbf{2}^{<\omega}$. However, since F is partial, we have to add the condition $u \in \mathit{dom}(F)$ which is $\Sigma^0_1 \wedge \Pi^0_1$ (cf. Proposition 2.4). Whence the constraint $n \geq 2$. ⊣

§4. **Definability in $\mathfrak{P}(\mathbb{N})$ w.r.t. Scott arithmetical hierarchy.** Tables 2, 3 give the level in Scott arithmetical hierarchy of diverse families $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ or $\subseteq \mathfrak{P}(\mathbb{N}^d)^\ell$. They are referred to in Propositions 4.1, 4.7. Let's recall that Scott definability in $\mathfrak{P}(\mathbb{N})$ is not the same as definability in the Cantor space $\mathbf{2}^\omega$, cf. Remark 3.3.

4.1. Simple set theoretic predicates on $\mathfrak{P}(\mathbb{N})$.

PROPOSITION 4.1 (Set theoretic predicates on $\mathfrak{P}(\mathbb{N})$). Table 2 gives the level in the Scott arithmetical hierarchy of some families $\mathscr{O} \subset \mathfrak{P}(\mathbb{N})$ (or $\mathfrak{P}(\mathbb{N}^2)$ or $\mathfrak{P}(\mathbb{N})^2$) defined by some set theoretic properties involving fixed parameters $A \subseteq \mathbb{N}$ and $p \in \mathbb{N}$ subject to specified restrictions.

When $\mathscr{O}$ is stated to be Scott Σ^0_n (resp. Π^0_n or Π^1_1) then it is not Scott Π^0_n (resp. Σ^0_n or Σ^1_1).

PROOF. The stated complexities are easily obtained using Proposition 3.4. Let's look at the negative assertion of the Proposition. The Σ^0_1 and Π^0_1 cases (lines 1a, 2a, 4a, 6b) are easy to handle since the involved $\mathscr{O}$'s are obviously not Scott clopen. Similarly, for the $\Sigma^0_1 \wedge \Pi^0_1$ case (line 4b), observe that $\mathscr{O}$ is neither open nor closed. The Π^1_1 case (line 19) is also easy since Scott Π^1_1 coincides with Π^1_1 in the sense of $\mathbf{2}^\omega$ and the result is well-known in the Cantor space framework.

Using Theorem 3.6, we see that if U is total computable $U : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ or $U : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ and $\mathscr{O}$ is Scott Σ^0_n then $U^{-1}(\mathscr{O})$ is Σ^0_n in $\mathbf{2}^{<\omega}$, so that the left cut of the real $\mu(U^{-1}(\mathscr{O}))$ is Σ^0_n. The same is true if U is self-delimited partial computable $U : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ and $n \geq 2$.

Now, an $(n+1)$-random real cannot have Σ^0_n left cut. Thus, the randomness results stated in Table 1 and proved in Theorems 9.4, 9.7 and 9.11 (their proofs do not depend on the negative assertion of the present Proposition) imply the remaining wanted negative results, except that for line 4d. If $\{A\}$ is Scott Σ^0_2 then it is a countable union of Scott $\Sigma^0_1 \wedge \Pi^0_1$ families and, being a singleton family, it is equal to one of these families, hence it is Scott $\Sigma^0_1 \wedge \Pi^0_1$. Finally, observe that $\{A\}$ is Scott $\Sigma^0_1 \wedge \Pi^0_1$ if and only if A is finite. In fact, let $\mathscr{X} = (\bigcup_{C \in \mathscr{C}} \mathscr{B}_C) \cap (\bigcap_{D \in \mathscr{D}} \overline{\mathscr{B}_D})$ where $\mathscr{C}, \mathscr{D}$ are c.e. subsets of $\mathfrak{P}_{<\omega}(\mathbb{N})$. If A is infinite and $A \in \mathscr{X}$ and $C \in \mathscr{C}$ is such that $C \subset A$ then any subset of A which contains C is also in $\mathscr{X}$, so that $\mathscr{X}$ cannot be equal to $\{A\}$. ⊣

4.2. Iterating operators *FIN* and *COF*.

DEFINITION 4.2 (Operators *FIN* and *COF*). 1. We denote by *FIN* and *COF* the operators $\mathfrak{P}(\mathbb{N}^d) \to \mathfrak{P}(\mathbb{N}^{d+1})$ such that, for all $\mathscr{X} \subseteq \mathfrak{P}(\mathbb{N}^d)$,

$$FIN(\mathscr{X}) = \{X \subseteq \mathbb{N}^{d+1} \mid \{i \mid X_i \in \mathscr{X}\} \text{ is finite}\},$$
$$COF(\mathscr{X}) = \{X \subseteq \mathbb{N}^{d+1} \mid \{i \mid X_i \in \mathscr{X}\} \text{ is cofinite}\}$$

where X_i denotes the set $X_i = \{(i_1, \ldots, i_k) \mid (i, i_1, \ldots, i_k) \in X\}$.

2. To any non empty word w in the alphabet $\{FIN, COF\}$ we associate a family $Set(w) \subseteq \mathfrak{P}(\mathbb{N}^{|w|})$ by the following induction:

$$Set(FIN) = \{X \subseteq \mathbb{N} \mid X \text{ is finite}\}, \qquad Set(FIN^\frown w) = FIN(Set(w)),$$
$$Set(COF) = \{X \subseteq \mathbb{N} \mid X \text{ is cofinite}\}, \qquad Set(COF^\frown w) = COF(Set(w)).$$

PROPOSITION 4.3. 1. If $n \geq 1$ and $\mathscr{X} \subseteq \mathfrak{P}(\mathbb{N}^d)$ is Scott Σ^0_n in $\mathfrak{P}(\mathbb{N}^d)$) then $FIN(\mathscr{X})$ and $COF(\mathscr{X})$ are respectively Scott Σ^0_{n+1} and Σ^0_{n+2} in $\mathfrak{P}(\mathbb{N}^{d+1})$.

2. Let $w = w_1 \ldots w_{i+j}$ be a non empty word in the alphabet $\{FIN, COF\}$ containing i occurrences of *FIN* and j occurrences of *COF*. Then $Set(w)$ is Scott Σ^0_{i+2j+1} in $\mathfrak{P}(\mathbb{N}^{|w|})$.

PROOF. 1. Observe that $FIN(\mathscr{X}) = \bigcup_{i\in\mathbb{N}} \bigcap_{j\geq i}\{X \mid X_j \notin \mathscr{X}\}$, and $COF(\mathscr{X}) = \bigcup_{i\in\mathbb{N}} \bigcap_{j\geq i}\{X \mid X_j \in \mathscr{X}\}$.

2. We argue by induction on $|w|$. *Initial step*: $|w| = 1$. We already know that $Set(FIN)$ is Scott Σ^0_2 in $\mathfrak{P}(\mathbb{N})$ (cf. Table 2 line 18). As for $Set(COF)$, observe that $\{X \mid X \text{ is cofinite}\} = \bigcup_{n\in\mathbb{N}}\{X \mid X \supseteq I_n\}$ where $I_n = \{i \mid i \geq n\}$. *Inductive step.* Apply point 1. ⊣

4.3. Many-one hardness of index sets for *FIN*, *COF* iterations. Simple variations of known many-one completeness of index sets of finite and cofinite sets apply to iterations of the *FIN*, *COF* operators (cf. §4.2).

PROPOSITION 4.4. 1. Let $n, d \geq 1$ and $\mathscr{C} \subseteq \mathbb{N}^d$. If $index(\mathscr{C})$ is Σ^0_n-complete then $index(FIN(\mathscr{C}))$ and $index(COF(\mathscr{C}))$ are respectively Σ^0_{n+1} and Σ^0_{n+2}-complete.

2. Let $w = w_1 \dots w_{i+j}$ be a word in the alphabet $\{FIN, COF\}$ containing i occurrences of *FIN* and j occurrences of *COF*. Then $index(Set(w))$ (cf. Definition 4.2) is Σ^0_{i+2j+1}-complete.

PROOF. Point 2 is immediate from point 1 and Proposition 4.3 for lines 6a and 8 of Table 2.

Hardness of $index(FIN(\mathscr{C}))$.Let $X \subseteq \mathbb{N}$ be Σ^0_{n+1}. There is some Σ^0_n set R such that

$$\begin{aligned} x \in X &\Leftrightarrow \exists i\ \neg R(x,i) \\ &\Leftrightarrow \exists i\ \exists j < i\ \neg R(x,j) \\ &\Leftrightarrow \exists i\ \neg(\forall j < i\ R(x,j)) \\ &\Leftrightarrow \{i : \forall j < i\ R(x,j)\} = [0,m] \text{ for some } m, \\ x \notin X &\Leftrightarrow \{i : \forall j < i\ R(x,j)\} = \mathbb{N}. \end{aligned}$$

Now, $\{(x,i) \mid \forall j < i\ R(x,j)\}$ is still Σ^0_n, so that the hypothesis on $\mathscr{C}$ insures that there is a computable map $\varphi : \mathbb{N}^2 \to \mathbb{N}$ such that, for all x, i,

$$\forall j < i\ R(x,j) \ \Leftrightarrow\ \varphi(x,i) \in index(\mathscr{C}) \ \Leftrightarrow\ W_{\varphi(x,i)} \in \mathscr{C}$$

Let $f : \mathbb{N} \to \mathbb{N}$ be computable such that $W_{f(x)} = \{(i,y) \mid y \in W_{\varphi(x,i)}\}$. Then $(W_{f(x)})_i = W_{\varphi(x,i)}$ and

$$\begin{aligned} x \in X &\Rightarrow \{i : (W_{f(x)})_i \in \mathscr{C}\} = [0,m] \text{ for some } m, \\ x \notin X &\Rightarrow \{i : (W_{f(x)})_i \in \mathscr{C}\} = \mathbb{N}. \end{aligned}$$

In particular, $X = f^{-1}(FIN(\mathscr{C}))$.

Hardness of $index(COF(\mathscr{C}))$. Let $X \subseteq \mathbb{N}$ be Σ^0_{n+2}. The first $\forall\exists$ alternation in the Π^0_{n+2} definition of $\mathbb{N} \setminus X$ can be replaced by a $\exists^\infty$ quantifier (cf. Kreisel & Shoenfield & Wang, [12] 1960, or Rogers' book [13] p.329). Thus, there is some Σ^0_n set R such that, for all x, $x \notin X \ \Leftrightarrow\ \exists^\infty i\ \neg R(x,i)$. The hypothesis on $\mathscr{C}$ insures that there is a computable map $\varphi : \mathbb{N}^2 \to \mathbb{N}$ such that, for all x, i, $R(x,i) \ \Leftrightarrow\ \varphi(x,i) \in index(\mathscr{C}) \ \Leftrightarrow\ W_{\varphi(x,i)} \in \mathscr{C}$. Let $f : \mathbb{N} \to \mathbb{N}$ be computable such that $W_{f(x)} = \{(i,y) \mid y \in W_{\varphi(x,i)}\}$. Then $(W_{f(x)})_i = W_{\varphi(x,i)}$ and

$$\begin{aligned} x \notin X &\Rightarrow \{i : (W_{f(x)})_i \notin \mathscr{C}\} \text{ is infinite}, \\ x \in X &\Rightarrow \{i : (W_{f(x)})_i \notin \mathscr{C}\} \text{ is finite} \\ &\Rightarrow \{i : (W_{f(x)})_i \in \mathscr{C}\} \text{ is cofinite}. \end{aligned}$$

Thus, $X = f^{-1}(COF(\mathscr{C}))$. ⊣

4.4. From index sets to Scott definability of families of c.e. sets. In this section we give a convenient tool to get the level in the Scott arithmetical hierarchy of some families of c.e. sets.

PROPOSITION 4.5 (From index sets to Scott definability). Let $\mathscr{O} \subseteq \Sigma^0_1(\mathbb{N})$ and $index(\mathscr{O}) = \{e \in \mathbb{N} \mid W_e \in \mathscr{O}\}$ (cf. Notation 1.1) be the index set of $\mathscr{O}$.

1. If $\mathscr{O} \subseteq \Sigma^0_1(\mathbb{N})$ has Σ^0_k index set then $\mathscr{O}$ is Scott $\Sigma^0_{\max(3,k)}$ in $\mathfrak{P}(\mathbb{N})$.
2. If $\mathscr{O} \subseteq \Sigma^0_1(\mathbb{N})$ has Π^0_k index set then $\mathscr{O}$ is Scott $\Sigma^0_3 \wedge \Pi^0_{\max(3,k)}$ in $\mathfrak{P}(\mathbb{N})$.
3. If $\mathscr{O} \subseteq \Sigma^0_1(\mathbb{N})$ has Π^1_1 index set then $\mathscr{O}$ is Scott Π^1_1 in $\mathfrak{P}(\mathbb{N})$.

PROOF. 1. First, observe that $\mathscr{W} = \{(X, e) \mid X = W_e\}$ is Scott Π^0_2. In fact, $X \neq W_e$ if and only if $\exists x\ ((x \in X \wedge x \notin W_e) \vee (x \notin X \wedge x \in W_e))$, so that $\mathscr{W}$ is the complement of the projection over $x \in \mathbb{N}$ of a union of two $\Sigma^0_1 \wedge \Pi^0_1$ families, hence is Scott Π^0_2. Now, $X \in \mathscr{O}$ if and only if $\exists e\ (X = W_e \wedge e \in index(\mathscr{O}))$. Thus, $\mathscr{O}$ is the projection over $e \in \mathbb{N}$ of the intersection of the Scott Π^0_2 family $\mathscr{W}$ with $\mathfrak{P}(\mathbb{N}) \times index(\mathscr{O})$, which is $\Sigma^0_{\max(3,k)}$ in $\mathfrak{P}(\mathbb{N})$ if $index(\mathscr{O})$ is Σ^0_k.

2. We also have $X \in \mathscr{O}$ if and only if $(\exists e\ X = W_e) \wedge \forall e\ (X \neq W_e \vee e \in index(\mathscr{O}))$ which shows that $\mathscr{O}$ is Scott $\Sigma^0_3 \wedge \Pi^0_{\max(3,k)}$. Point 3 is similar. ⊣

4.5. Some predicates on $\mathfrak{P}(\mathbb{N})$ from computability theory. First, we recall some classical definitions (cf. Soare's book [17]).

DEFINITION 4.6.
- X is low if its jump X' has the same Turing degree as $\emptyset'$. If X is low c.e., we have $\Sigma^{0,X}_1 \subseteq \Delta^0_2$ and, for $n \geq 2$, $\Sigma^{0,X}_n = \Sigma^0_2$ and $\Pi^{0,X}_n = \Pi^0_n$.
- X is high if its jump X' has the same Turing degree as $\emptyset''$. If X is high c.e., we have $\Sigma^{0,X}_n = \Sigma^0_{n+1}$ and $\Pi^{0,X}_n = \Pi^0_{n+1}$ for all $n \geq 2$.
- X is n-low if its n-th jump $X^{(n)}$ has the same Turing degree as $\emptyset^{(n)}$.
- X is n-high if its n-th jump $X^{(n)}$ has the same Turing degree as $\emptyset^{(n+1)}$.
- X is simple if it is c.e. and its complement is infinite but contains no infinite c.e. set.
- X is maximal if it is c.e. and, for any c.e. set $Y \supseteq X$, either Y is cofinite or $Y \setminus X$ is finite.
- X is atomless if it is coinfinite c.e. with no maximal c.e. superset.

PROPOSITION 4.7 (Recursion theoretic predicates on $\mathfrak{P}(\mathbb{N})$). Table 3 gives the level in the Scott arithmetical hierarchy of some families $\mathscr{O} \subset \mathfrak{P}(\mathbb{N})$ (or $\mathfrak{P}(\mathbb{N})^2$) defined by some recursion theoretic properties involving a fixed parameter $A \subseteq \mathbb{N}$ subject to specified restrictions.

When $\mathscr{O}$ is stated to be Scott Σ^0_n (resp. Π^0_n) then it is not Scott Π^0_n (resp. Σ^0_n).

PROOF. Excepted for lines 14abc, all stated complexities are easily obtained using Proposition 3.4.

Case of line 14a. We adapt Yates argument for the $\Sigma^{0,A}_3$ character of the index set $\{e \mid W_e \leq_{Turing} A\}$ when A is c.e. (cf. Soare [17], p.242). Denote by $\{e\}^A$ the partial A-computable function $\mathbb{N} \to \mathbb{N}$ with code e. Then

$$X \leq_{Turing} A \ \Leftrightarrow\ \exists e\ (\{e\}^A \text{ is total } \wedge\ \forall x\ ((x \in X \Leftrightarrow \{e\}^A(x) = 1))).$$

Observe that the total character of $\{e\}^A$ is a $\Pi^{0,A}_2$ statement about e. Since A is low, this is also Π^0_2. To deal with the last part of the above equivalence, consider

	$\mathscr{O}$ **is the set of** X**'s such that**	**complex.**
9a	X is c.e.	Σ^0_3
9b	X is a boolean combination of c.e. sets	Σ^0_3
9c	Idem with X A-c.e. if A is low c.e.	Σ^0_3
10a	X is computable	Σ^0_3
10b	the complement of X is c.e.	Σ^0_3
11a	X is simple	$\Sigma^0_3 \wedge \Pi^0_3$
11b	X is c.e. not simple	Σ^0_3
12	X is maximal	Π^0_4
13	X is atomless	Π^0_5
14a	$X \leq_{Turing} A$ where A is low c.e.	Σ^0_3
14b	$X \equiv_{Turing} A$ where A is low c.e.	Σ^0_3
14c	Idem with the additional condition X c.e.	Σ^0_3
15a	$X \leq_{Turing} A$ where A is high c.e.	Σ^0_4
15b	$X \equiv_{Turing} A$ where A is high c.e.	Σ^0_4
15c	Idem with the additional condition X c.e.	Σ^0_4
16a	$X \leq_{Turing} Y$	Σ^0_4
16b	$X \equiv_{Turing} Y$	Σ^0_4
17a	X is c.e. and n-low	Σ^0_{n+3}
17b	X is c.e. and n-high	Σ^0_{n+4}

TABLE 3. Complexity of recursion theoretic families $\mathscr{O}$ in the Scott arithmetical hierarchy.

the computable predicate $S(x, u, e, y, z, t)$ which states that there is an oracular computation of machine e on input x which at step t outputs u, using the sole conditions that the oracle contains $f(y)$ and is disjoint from $f(z)$. The total character of $\{e\}^A$ allows to express $\{e\}^A(x) = 1$ in both forms

$$\exists y, z, t\ (S(x, 1, e, y, z, t) \wedge f(y) \subseteq A \wedge f(z) \cap A = \emptyset)$$
$$\forall y, z, y, u\ ((S(x, u, e, y, z, t) \wedge f(y) \subseteq A \wedge f(z) \cap A = \emptyset) \Rightarrow u = 1)$$

which are Σ^0_2 and Π^0_2 in $\mathbb{N}^2$ (A being a fixed c.e. set). Reporting in the above equivalence, we see that the family $\mathscr{O}$ of line 14a is Scott Σ^0_3.

Case of line 14*b*. Again, we adapt Yates argument for the $\Sigma^{0,A}_3$ character of the index set $\{e \mid W_e \equiv_{Turing} A\}$ when A is c.e. (cf. Soare [17], p.242):

$$X \equiv_{Turing} A \Leftrightarrow \exists e, i\ (\{e\}^A \text{ and } \{i\}^{W_e^A} \text{ are total}$$
$$\wedge\ \forall x\ ((x \in X \Leftrightarrow \{e\}^A(x) = 1) \wedge (x \in A \Leftrightarrow \{i\}^X(x) = 1)))$$

Observe that the total character of $\{e\}^A$ and $\{i\}^{W_e^A}$ is a $\Pi^{0,A}_2$ statement about e, i. Since A is low, this is also Π^0_2.

Case of line 14*c*. Immediate from lines 9a and 14ab.

Negative assertion of the Proposition. The randomness argument used to prove the similar assertion of Proposition 4.1 works for all lines of Table 3, cf. the randomness results stated in Table 1 and proved in Theorems 9.4, 9.7 and 9.11 (their proofs do not depend on the negative assertion of the present Proposition). $\dashv$

§5. Three notions of hardness for subsets of $\mathfrak{P}(\mathbb{N})$ or $\mathfrak{P}(\mathbb{N}^d)^\ell$. Effective Wadge hardness, second order many-one hardness and special hardness are the basic tools to obtain randomness results, and they arise from the three different sorts of computable maps considered in §2. In this section we present their definitions, while in §6, §7 and §8 we develop characterizations for families of subsets of $\mathfrak{P}(\mathbb{N})$ at Scott levels Σ^0_n and Π^0_n.

Again, to simplify notations, we reduce to the case $\mathfrak{P}(\mathbb{N})$, the general case $\mathfrak{P}(\mathbb{N}^d)^\ell$ being trivial extension. Recall that we use $\mu(\mathscr{X})$ to denote the Lebesgue measure of a subset $\mathscr{X}$ of the Cantor space $\mathbf{2}^\omega$ of all infinite binary words of length ω.

5.1. Effective Wadge hardness.

Definition 5.1 (Effective Wadge hardness). 1. The set $\mathscr{A} \subseteq \mathbf{2}^\omega$ is *effectively Wadge reducible* to $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ if there exists a computable total map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ such that $\mathscr{A} = F^{-1}(\mathscr{O})$.

2. Effective *almost everywhere* Wadge reducibility and effective *measure* Wadge reducibility are respectively obtained by weakening equality to equality up to a set of measure zero and to equality of measures up to a computable inversible linear transformation. I.e. respectively asking for $\mu(\mathscr{A}\Delta F^{-1}(\mathscr{O})) = 0$, $\mu(F^{-1}(\mathscr{O})) = a\mu(\mathscr{A}) + b$, with a, b computable, $a \neq 0$, where $X\Delta Y = (X \setminus Y) \cup (Y \setminus X)$ is the symmetric difference of the sets X, Y.

   We write $\mathscr{A} \leq^{\mathit{eff}}_W \mathscr{O}$, $\mathscr{A} \leq^{a.e.}_W \mathscr{O}$, $\mathscr{A} \leq^{\mathit{meas}}_W \mathscr{O}$.

3. The set $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ is *effectively Wadge hard* for a class $\mathscr{C}$ of subsets of $\mathbf{2}^\omega$ if $\mathscr{A} \leq^{\mathit{eff}}_W \mathscr{O}$ for all $\mathscr{A} \in \mathscr{C}$. Effective almost everywhere Wadge hardness and effective measure Wadge hardness are defined similarly.

Remark 5.2. Since computable maps $\mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ are continuous, the above notions of *effective Wadge reducibility* and *effective Wadge hardness* are effectivizations of the classical topological notions of reducibility and hardness introduced by Wadge [19].

Remark 5.3. The *almost everywhere Wadge* and *measure* Wadge variants happen to be pertinent conditions for some of our applications (cf. Theorems 7.7 and 9.2). In particular, the family of c.e. subsets of $\mathbb{N}$ will be proved to be effective almost everywhere Wadge hard for $\Sigma^0_3(\mathbf{2}^\omega)$ but not effective Wadge hard, cf. Theorem 7.7.

5.2. Second order many-one hardness. We extend the classical many-one reducibility between sets of integers or finite words to a second order context using computable total maps $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$.

Definition 5.4 (Second order many-one hardness). 1. The set $A \subseteq \mathbf{2}^{<\omega}$ is *second order many-one reducible* to $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ (written $A \leq_m \mathscr{O}$) if there exists a computable total map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ such that $A = F^{-1}(\mathscr{O})$.

2. *Open* (resp. *almost everywhere*, resp. *measure*) second order many-one reducibility is obtained by weakening equality $A = F^{-1}(\mathscr{O})$ to equality of the open subsets of $\mathbf{2}^\omega$ associated to the restrictions of A and $F^{-1}(\mathscr{O})$ to length $\geq k$ words (resp. equality up to a set of measure zero, resp. equality of measures up to a computable inversible linear transformation). I.e. respectively asking for $A\mathbf{2}^\omega = F^{-1}(\mathscr{O})\mathbf{2}^\omega$, $\mu(A\mathbf{2}^\omega\Delta F^{-1}(\mathscr{O})\mathbf{2}^\omega) = 0$, $\mu(F^{-1}(\mathscr{O})\mathbf{2}^\omega) = a\mu(A\mathbf{2}^\omega) + b$ with a, b with computable, $a \neq 0$.

   We write $A \leq_m \mathscr{O}$, $A \leq^{\mathit{open}}_m \mathscr{O}$, $A \leq^{a.e.}_m \mathscr{O}$ and $A \leq^{\mathit{meas}}_m \mathscr{O}$.

3. The set $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ is *second order many-one hard* for a class $\mathscr{C}$ of subsets of $\mathbf{2}^{<\omega}$ if $A \leq_m \mathscr{O}$ for all $A \in \mathscr{C}$. Open, almost everywhere and measure second order many-one hardness are defined similarly.

5.3. Special hardness. Replacing in Definition 5.4 the condition $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ *is total computable* by the condition *F is self-delimited partial computable*, we get the notion of special reducibility and its variants.

Definition 5.5 (Special hardness). 1. The set $A \subseteq \mathbf{2}^{<\omega}$ is *special reducible* to $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$, if there exists a self-delimited partial computable map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ such that $A = F^{-1}(\mathscr{O})$.

2. *Open* (resp. *almost everywhere*, resp. *measure*) special reducibility is obtained by weakening equality $A = F^{-1}(\mathscr{O})$ to equality of the associated open subsets of $\mathbf{2}^\omega$ (resp. equality up to a set of measure zero, resp. equality of measures up to a computable inversible linear transformation). I.e. asking for $A\mathbf{2}^\omega = F^{-1}(\mathscr{O})\mathbf{2}^\omega$ (resp. $\mu(A\mathbf{2}^\omega \Delta F^{-1}(\mathscr{O})\mathbf{2}^\omega) = 0$, resp. $\mu(F^{-1}(\mathscr{O})\mathbf{2}^\omega) = a\mu(A\mathbf{2}^\omega) + b$ for some computable a, b with $a \neq 0$).
   We write $A \leq_{\mathrm{sp}} \mathscr{O}$, $A \leq_{\mathrm{sp}}^{open} \mathscr{O}$, $A \leq_{\mathrm{sp}}^{a.e.} \mathscr{O}$ and $A \leq_{\mathrm{sp}}^{meas} \mathscr{O}$.
3. The set $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ is special hard for a class $\mathscr{C}$ of subsets of $\mathbf{2}^{<\omega}$ if $A \leq_{spec} \mathscr{O}$ for all $A \in \mathscr{C}$. Special almost everywhere hardness and special measure hardness are defined similarly.

Remark 5.6. Clearly, special reducibility of A to $\mathscr{O}$ implies that A is included in some prefix-free $\Sigma^0_1 \wedge \Pi^0_1$ subset of $\mathbf{2}^{<\omega}$ (namely, the domain of F, cf. Proposition 2.4). As we shall see in §8.2, this forbids the existence of Σ^0_n or Π^0_n open special hard families $\mathscr{O}$ for $n \geq 3$.

5.4. Hardness and complementation.

Proposition 5.7. Let $\mathscr{A} \subseteq \mathbf{2}^\omega$ and $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$.

1. $\mathscr{A} \leq_W^{eff} \mathscr{O}$ if and only if $\overline{\mathscr{A}} \leq_W^{eff} \overline{\mathscr{O}}$.
2. $\mathscr{O}$ is effectively Wadge hard for $\mathscr{C} \subseteq \mathfrak{P}(\mathbf{2}^\omega)$ if and only if $\overline{\mathscr{O}}$ is effectively Wadge hard for the class $\check{\mathscr{C}} = \{\overline{X} \mid X \in \mathscr{C}\}$.
3. Points 1 and 2 also hold with almost everywhere and with measure reducibility hardness.

Proof. For points 1,2, use commutation of F^{-1} with complementation. For 3, use the identity $\mathscr{A} \Delta \mathscr{B} = \overline{\mathscr{A}} \Delta \overline{\mathscr{B}}$. ⊣

Proposition 5.8. Let $A \subseteq \mathbf{2}^{<\omega}$ and $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$.

1. $A \leq_m \mathscr{O}$ if and only if $\overline{A} \leq_m \overline{\mathscr{O}}$.
2. $\mathscr{O}$ is second order many-one hard for $\mathscr{C} \subseteq \mathfrak{P}(\mathbf{2}^{<\omega})$ if and only if $\overline{\mathscr{O}}$ is second order many-one hard for the class $\check{\mathscr{C}} = \{\overline{X} \mid X \in \mathscr{C}\}$.
3. Points 1, 2 fail for open, almost everywhere and measure second order many-one hardness.

Proof. For points 1,2, use commutation of F^{-1} with complementation over $\mathbf{2}^{\geq k}$. The basic reason for point 3 is that $A\mathbf{2}^\omega$ and $\overline{A}\mathbf{2}^\omega$ may be non complementary sets (they may even be equal sets). Counterexamples proving point 3 are given by Theorem 5.11 and Proposition 5.16 below. ⊣

Remark 5.9. Proposition 5.8 is always false with special reducibility since F is partial and $F^{-1}(\overline{\mathscr{O}}) = dom(F) \setminus F^{-1}(\mathscr{O})$ is not the complement of $F^{-1}(\mathscr{O})$. In fact, $\mathscr{O} = \mathfrak{P}(\mathbb{N})$ is open special hard for Σ^0_2 (cf. Proposition 8.1) whereas its complement special reduces the sole empty set.

5.5. Open many-one hardness and al.: from Π^0_n to Σ^0_{n+1}.

Lemma 5.10. Let $n \geq 1$. For any Σ^0_n set $A \subseteq \mathbf{2}^{<\omega}$ there exists some prefix-free Π^0_{n-1} set $B \subseteq \mathbf{2}^{<\omega}$ such that $A\mathbf{2}^{\omega} = B\mathbf{2}^{\omega}$.

Proof. *Case $n = 1$.* Well-known fact (where Π^0_0 means computable). Recall the argument: (1) replace a word $u \in A$ by all words in $u\mathbf{2}^k$ for some k so as to transform some fixed computable enumeration of A into a length non decreasing computable enumeration of A_0 such that $A\mathbf{2}^{\omega} = A_0\mathbf{2}^{\omega}$, (2) remove from A_0 any word which has some prefix already enumerated.

Getting B Δ^0_n. Relativizing case $n = 1$ to oracle $\emptyset^{(n)}$, we get the statement of the Lemma with B Δ^0_n.

Case $n = 2$. We have just observed that one can reduce to the case A is prefix-free (and also Δ^0_2 but this will be of no use). Let $A = \{u \mid \exists x\ \forall y\ R(x, y, u)\}$ where $R \subseteq \mathbb{N}^2 \times \mathbf{2}^{<\omega}$ is computable. The intuition for the following definition is to extend any $u \in A$ to all v with length coding the following triple of integers:

(i) the length of u (in order to recover u as a prefix of v).
(ii) the value of the least x such that $\forall y\ R(x, y, u)$.
(iii) the value of the least z such that $\forall s < x\ \exists y \leq z\ \neg R(s, y, u)$.

Letting $(\pi^3_1, \pi^3_2, \pi^3_3) : \mathbb{N} \to \mathbb{N}^3$ be Cantor computable bijection. Recall that $\pi^3_i(t) \leq t$ for all t. Define B as follows:

$$\begin{aligned} B = \{v \in \mathbf{2}^{<\omega} \mid\ & \forall y\ R(\pi^3_2(|v|), y, v \restriction \pi^3_1(|v|)) \\ & \wedge \forall x < \pi^3_2(|v|)\ \exists y \leq \pi^3_3(|v|)\ \neg\ R(x, y, v \restriction \pi^3_1(|v|)) \\ & \wedge \exists x < \pi^3_2(|v|)\ \forall y < \pi^3_3(|v|)\ \ R(x, y, v \restriction \pi^3_1(|v|))\}. \end{aligned}$$

Clearly, B is Π^0_1 definable. Let's see that B is prefix-free. Suppose $v \leq_{\text{pref}} w$ and v, w are both in B. Then $v \restriction \pi^3_1(|v|)$ and $w \restriction \pi^3_1(|w|)$ are both in A. Since they are prefix comparable (as are v and w) and A has been supposed prefix-free they are equal. Let $u = v \restriction \pi^3_1(|v|) = w \restriction \pi^3_1(|w|)$. Now, $\pi^3_2(|v|)$ and $\pi^3_2(|w|)$ are both equal to the least x such that $\forall y\ R(x, y, u)$. Similarly, $\pi^3_3(|v|)$ and $\pi^3_3(|w|)$ are both equal to the least z such that $\forall x' < x\ \exists y \leq z\ \neg R(x', y, u)$. Thus, $|v| = |w|$ and condition $v \leq_{\text{pref}} w$ yields equality $v = w$. Let's see that $A\mathbf{2}^{\omega} = B\mathbf{2}^{\omega}$. For inclusion $B\mathbf{2}^{\omega} \subseteq A\mathbf{2}^{\omega}$, observe that any $v \in B$ extends some $u \in A$. Now, suppose $u \in A$ and let k code $|u|$, x and z as in conditions (i), (ii), (iii) above. Then all extensions of u with length k are in B. This insures $A\mathbf{2}^{\omega} \subseteq B\mathbf{2}^{\omega}$.

Case $n \geq 3$. Relativize case $n = 2$ to oracle $\emptyset^{(n-1)}$. ⊣

As an immediate corollary, we get

Theorem 5.11. *If $\mathscr{O}$ is open second order many-one hard for $\Pi^0_n(\mathbf{2}^{<\omega})$ then it is also open second order many-one hard for $\Sigma^0_{n+1}(\mathbf{2}^{<\omega})$. Idem with almost everywhere and with measure hardness. Idem with special hardness.*

Proof. Given a Σ^0_{n+1} set A, apply Π^0_n open (resp. a.e., resp. measure) many-one hardness to a Π^0_n set B such that $A\mathbf{2}^{\omega} = B\mathbf{2}^{\omega}$. ⊣

5.6. Relations between the three notions of hardness. The next Proposition characterizes second order many-one hardness in terms of effective Wadge hardness.

PROPOSITION 5.12. Let $\mathscr{C} \subseteq \mathfrak{P}(\mathbf{2}^{<\omega})$ be a class of sets such that if $X \in \mathscr{C}$ then $\theta(X) \in \mathscr{C}$ for every computable injective $\theta : \mathbf{2}^{<\omega} \to \mathbf{2}^{<\omega}$ which has computable range (for example $\mathscr{C} = \Sigma^0_n(\mathbf{2}^{<\omega})$ or $\mathscr{C} = \Pi^0_n(\mathbf{2}^{<\omega})$). Consider the following conditions:

i. $\mathscr{O}$ is effectively Wadge hard for the class of all sets $X\mathbf{2}^{\omega}$ where $X \in \mathscr{C}$ is a subset of some infinite computable prefix-free set of words.
ii. $\mathscr{O}$ is second order many-one hard for $\mathscr{C}$.

Then,

1. i $\Rightarrow$ ii.
2. ii $\Rightarrow$ i holds if $\emptyset \notin \mathscr{O}$.

PROOF. 1. Let $B \subseteq \mathbf{2}^{<\omega}$ be an infinite computable prefix-free set of words. Then, $\mathscr{O}$ is effectively Wadge hard for the class of all sets $X\mathbf{2}^{\omega}$ where $X \in \mathscr{C}$ is a subset of B. Fix some computable bijection θ between $\mathbf{2}^{<\omega}$ and B. Let $A \in \mathscr{C}$. Then $\theta(A) \in \mathscr{C}$. Let $F : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$ be a total computable map such that $\theta(A)\mathbf{2}^{\omega} = F^{-1}(\mathscr{O})$ and define $G : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ as $G(u) = F(\theta(u)0^{\omega})$. G is total computable and $G(u) \in \mathscr{O} \Leftrightarrow \theta(u)0^{\omega} \in F^{-1}(\mathscr{O}) \Leftrightarrow \theta(u)0^{\omega} \in \theta(A)\mathbf{2}^{\omega} \Leftrightarrow \theta(u) \in \theta(A) \Leftrightarrow u \in A$ (for the third equivalence, use the fact that the range B of θ is prefix-free).

2. Let $X \in \mathscr{C}$ be a subset of the computable prefix-free $Z \subseteq \mathbf{2}^{<\omega}$. Let $\mathscr{M}$ be a Turing machine computing a total computable $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ such that $X = F^{-1}(\mathscr{O})$. Consider the Turing machine $\mathscr{T}$ which, on input $\alpha \in \mathbf{2}^{\omega}$, outputs nothing until it has read a prefix of its input α lying in Z. If and when such a prefix $\alpha \upharpoonright t$ appears, $\mathscr{T}$ starts ouputting $F(\alpha \upharpoonright t)$ as $\mathscr{M}$ does. Clearly, $\mathscr{T}$ computes a total map $G : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$. If $\alpha \notin Z\mathbf{2}^{\omega}$ then $G(\alpha) = \emptyset \notin \mathscr{O}$. If $\alpha = u\beta$ with $u \in Z$ then $\alpha \in X\mathbf{2}^{\omega} \Leftrightarrow u \in X \Leftrightarrow F(u) \in \mathscr{O} \Leftrightarrow G(\alpha) \in \mathscr{O}$. Thus, $X\mathbf{2}^{\omega} = G^{-1}(\mathscr{O})$, as wanted. ⊣

REMARK 5.13. Proposition 5.12 can be improved replacing point 2 with:

2. ii $\Rightarrow$ i holds if $\emptyset \notin \mathscr{O}$ or $\exists C$ (C is c.e. not in $\mathscr{O}$ $\wedge$ $\forall X \in \mathfrak{P}(\mathbf{2}^{<\omega})\ \forall Y \in \mathfrak{P}_{<\omega}(C)\ (X \in \mathscr{O} \Leftrightarrow X \cup Y \in \mathscr{O})$).

The given proof adapts requiring that machine $\mathscr{T}$, which computes the wanted reduction G, first outputs the elements of a computable enumeration of C until it reads prefix of α lying in Z. If such prefix exists, $G(\alpha)$ is $F(u)$ augmented with finitely many elements of C. The extra hypothesis insures that $G(\alpha) \in \mathscr{O} \Leftrightarrow F(u) \in \mathscr{O}$.

To get the expected Corollary 5.15, let's look at the relations between the complexity of $A \subseteq \mathbf{2}^{<\omega}$ and that of $A\mathbf{2}^{\omega}$.

PROPOSITION 5.14. Let $n \geq 1$ and $A \subseteq \mathbf{2}^{<\omega}$. Then,

i. A is $\Sigma^0_n \Rightarrow A\mathbf{2}^{\omega}$ is Σ^0_n,
ii. A is $\Pi^0_n \Rightarrow A\mathbf{2}^{\omega}$ is Π^0_n if $\exists B$ prefix-free Σ^0_{n-1} $A \subseteq B$,
iii. $A\mathbf{2}^{\omega}$ is $\Sigma^0_n \Rightarrow A$ is Σ^0_n if $\exists B$ prefix-free Σ^0_n $A \subseteq B$,
iv. $A\mathbf{2}^{\omega}$ is $\Pi^0_n \Rightarrow A$ is Π^0_n if $\exists B$ prefix-free Π^0_n $A \subseteq B$.

PROOF. For i and ii, observe that $\alpha \in A\mathbf{2}^{\omega}$ if and only if $\exists n\ \alpha \upharpoonright n \in A$ if and only if $(\exists n\ \alpha \upharpoonright n \in B) \wedge \forall n\ (\alpha \upharpoonright n \in B \Rightarrow \alpha \upharpoonright n \in A)$. As for iii and iv, observe that $u \in A$ if and only if $u \in B \wedge u0^{\omega} \in A\mathbf{2}^{\omega}$. ⊣

COROLLARY 5.15. If $\mathcal{O} \subseteq \mathfrak{P}(\mathbb{N})$ is effectively Wadge hard for $\Sigma^0_n(\mathbf{2}^\omega)$ then it is also second order many-one hard for $\Sigma^0_n(\mathbf{2}^{<\omega})$. Idem with Π^0_n.

PROOF. Use points i, ii of Proposition 5.14 and i $\Rightarrow$ iii in Proposition 5.12. $\dashv$

5.7. Hardness and the arithmetical hierarchy. Neither Wadge duality theorem nor Wadge hardness theorem [19] apply to effective Wadge, second order many-one, nor special reducibility. Nevertheless, the easy direction of the hardness theorem does hold.

PROPOSITION 5.16. Let $n \geq 1$ and B be an infinite computable prefix-free set of words. Let $\mathscr{C}$ be the class of Σ^0_n (resp. Π^0_n) subsets of B.

1. If $\mathcal{O} \subseteq \mathfrak{P}(\mathbb{N})$ is effectively Wadge hard for $\Sigma^0_n(\mathbf{2}^\omega)$ (resp. $\Pi^0_n(\mathbf{2}^\omega)$) then $\mathcal{O}$ is not Scott Π^0_n (resp. not Scott Σ^0_n).
   Idem if $\mathcal{O}$ is effectively Wadge hard for the class of subsets $X\mathbf{2}^\omega$ such that $X \in \mathscr{C}$.
2. If $\mathcal{O} \subseteq \mathfrak{P}(\mathbb{N})$ is second order open many-one hard for $\Sigma^0_n(\mathbf{2}^{<\omega})$ (resp. $\Pi^0_n(\mathbf{2}^{<\omega})$) then $\mathcal{O}$ is not Scott Π^0_n (resp. Σ^0_n).
   Idem if $\mathcal{O}$ is second order open many-one hard for $\mathscr{C}$.
3. If $n \geq 2$ and $\mathcal{O} \subseteq \mathfrak{P}(\mathbb{N})$ is open special hard for $\Sigma^0_n(\mathbf{2}^{<\omega})$ (resp. $\Pi^0_n(\mathbf{2}^{<\omega})$) then $\mathcal{O}$ is not Scott Π^0_n (resp. Σ^0_n).
   Idem if $\mathcal{O}$ is second order open many-one hard for $\mathscr{C}$.

PROOF. Proposition 5.14 shows that it suffices to prove the statements involving $\mathscr{C}$. We consider the sole case $\mathscr{C} = \Sigma^0_n(B)$, the case $\mathscr{C} = \Pi^0_n(B)$ is similar.

Case $\mathcal{O}$ is effectively Wadge hard for $X\mathbf{2}^\omega$'s such that $X \in \mathscr{C}$. Consider some Σ^0_n set $X \subseteq B$ which is not Π^0_n. If $\mathcal{O}$ were Scott Π^0_n then, by Theorem 3.6, $X\mathbf{2}^\omega$, being effectively Wadge reducible to $\mathcal{O}$, would be $\Pi^0_n(\mathbf{2}^\omega)$. Using Proposition 5.14, this would imply that X is Π^0_n, a contradiction.

Case $\mathcal{O}$ is second order open many-one hard for $\mathscr{C}$. Similar.

Case $\mathcal{O}$ is open special hard for $\mathscr{C}$. Idem but $n \geq 2$ is required in Theorem 3.6. $\dashv$

§6. Criteria for second order many-one hardness.

6.1. Second order many-one hardness at level 1.

PROPOSITION 6.1 (Hardness at level Σ^0_1). Let $\mathcal{O} \subseteq \mathfrak{P}(\mathbb{N})$ be Scott Σ^0_1. The following conditions are equivalent:

i. $\mathcal{O}$ is second order many-one hard for $\Sigma^0_1(\mathbf{2}^{<\omega})$,
ii. $\mathcal{O} \neq \emptyset$ and $\emptyset \notin \mathcal{O}$.

PROOF. i $\Rightarrow$ ii. If $\mathcal{O}$ were empty (resp. equal to $\mathfrak{P}(\mathbb{N})$) then $F^{-1}(\mathcal{O})$ would be empty (resp. equal to $\mathbf{2}^{<\omega}$) for all total $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$, contradicting i. Since $\mathcal{O}$ is Scott Σ^0_1 it is open for the Scott topology and condition $\mathcal{O} \neq \mathfrak{P}(\mathbb{N})$ is equivalent to $\emptyset \notin \mathcal{O}$.

ii $\Rightarrow$ i. Being Scott Σ^0_1 and non empty, $\mathcal{O}$ contains a basic open set $\mathscr{B}_A$ for some finite A. Since $\emptyset \notin \mathcal{O}$ we have $A \neq \emptyset$. Let X be $\Sigma^0_1(\mathbf{2}^{<\omega})$. Define a computable total map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ such that $F(u) = A$ if $u \in X$ and $F(u) = \emptyset$ otherwise. Since $A \in \mathcal{O}$ and $\emptyset \notin \mathcal{O}$, we have $F^{-1}(\mathcal{O}) = X$. $\dashv$

As a straightforward application, we get hardness results for the level 1 families of Table 2 (cf. Proposition 4.1).

PROPOSITION 6.2. The family $\{X \in \mathfrak{P}(\mathbb{N}) \mid X \supseteq A\}$ (line 1a of Table 2) is second order many-one hard for $\Sigma^0_1(\mathbf{2}^{<\omega})$ if $A \subseteq \mathbb{N}$ is finite non empty.

The families $\mathfrak{P}(A)$ with $A \neq \mathbb{N}$ Π^0_1 (line 2a), $\{\emptyset\}$ (line 4a) and $\{X \mid |X| \leq p\}$ (line 6b) from Table 2 are second order many-one hard for $\Pi^0_1(\mathbf{2}^{<\omega})$.

6.2. Computable sieves for $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$. To get a convenient sufficient condition for second order many-one and effective Wadge hardness at level Σ^0_2, we introduce the notion of sieve.

DEFINITION 6.3 (Computable sieve for $\mathscr{O}$). A computable sieve for $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ is a computable map $v : \mathbb{N}^{<\omega} \to \mathfrak{P}_{<\omega}(\mathbb{N})$ such that, denoting $\sigma\frown n$ the finite sequence obtained by appending $n \in \mathbb{N}$ to $\sigma \in \mathbb{N}^{<\omega}$,

i. For every $\sigma \in \mathbb{N}^{<\omega}$, the sequence $(v(\sigma\frown n))_{n\in\mathbb{N}}$ is monotone nondecreasing (w.r.t. set inclusion) and its union is in $\mathscr{O}$,
ii. For every $\varphi \in \mathbb{N}^{\omega}$, the sequence $(v(\varphi \upharpoonright n))_{n\in\mathbb{N}}$ is monotone nondecreasing and its union is not in $\mathscr{O}$.

The notion extends easily to $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N}^d)^\ell$ with componentwise set inclusion.

A co-sieve for $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ is a sieve for its complement $\overline{\mathscr{O}}$.

The chain condition below was already used for subsets of $\mathbf{2}^{\leq\omega}$ in Definition 1.15 of part I [5].

PROPOSITION 6.4. The following "*computable chain condition*" implies that $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ admits a computable sieve:

(chain) There exists a computable monotone non decreasing chain $(X_i)_{i\in\mathbb{N}} \in (\mathfrak{P}(\mathbb{N}))^\omega$ of sets in $\mathscr{O}$, the limit of which is not in $\mathscr{O}$.

PROOF. Let $v(\langle\rangle) = \emptyset$ and $v(\langle n_1, \dots, n_k\rangle) = X_k \cap \{0, \dots, k + n_k\}$. ⊣

6.3. Second order many-one hardness at level 2. We can now get a sufficient condition second order many-one hardness for Σ^0_2.

PROPOSITION 6.5. If $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ has a computable sieve (resp. co-sieve) then $\mathscr{O}$ is second order many-one hard for $\Sigma^0_2(\mathbf{2}^{<\omega})$ (resp. $\Pi^0_2(\mathbf{2}^{<\omega})$).

PROOF. Let $X \in \Sigma^0_2(\mathbf{2}^{<\omega})$ be such that $X = \{u \mid \exists i\ \forall j\ R(i, j, u)\}$ where R is computable. The second order many-one reduction $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ of X to $\mathscr{O}$ will be such that $F(u) = \bigcup_{t\in\mathbb{N}} v(h(u,t))$ where $h : \mathbf{2}^{<\omega} \times \mathbb{N} \to \mathbb{N}^{<\omega}$ is a computable map defined by the following induction: for $u \in \mathbf{2}^{<\omega}$, $t \in \mathbb{N}$,

$$h(u, 0) = \langle 0\rangle,$$
$$h(u, t+1) = \begin{cases} \sigma\frown(p+1) & \text{if } h(u) = \sigma\frown p \text{ and } R(|\sigma|, p, u) \text{ holds,} \\ \sigma\frown p\frown 0 & \text{otherwise.} \end{cases}$$

For $u \in \mathbf{2}^{<\omega}$, if $h(u,t) = \langle n_1, ..., n_k\rangle$ then

- $t = k - 1 + \sum_{i=1,\dots,k} n_i$
- $R(i, j, u)$ holds for $i \leq k$ and $j < n_i$
- $R(i, j, u)$ fails for $i < k$ and $j = n_i$.

Observe that the sequence $(h(u,t))_{t\in\mathbb{N}}$ is monotone increasing with respect to the lexicographic ordering on $\mathbb{N}^{<\omega}$.

Case $u \in X$. Let i be least such that $\forall j\ R(i,j,u)$ and $\sigma = \langle n_0,\dots,n_{i-1}\rangle$ where, for $\ell < i$, n_ℓ is least such that $R(\ell, n_\ell, u)$ fails. Then $h(u,t) = \sigma^\frown(t-p)$ for all $t \geq p = i-1+n_0+\dots+n_{i-1}$. Since v is a sieve for $\mathscr{O}$, the sequence $(v(h(u,t)))_{t\in\mathbb{N}}$ has limit in $\mathscr{O}$, i.e. $F(u) \in \mathscr{O}$.

Case $u \notin X$. Then the lengths of the $h(u,t)$'s are unbounded. Let $f \in \mathbb{N}^{<\omega}$ be such that the $f(t)^\frown 0$'s are the successive terms with last element 0 in the sequence $(h(u,n)_{n\in\mathbb{N}})$. By monotonicity, we have $\bigcup_{n\in\mathbb{N}} v(h(u,n)) = \bigcup_{t\in\mathbb{N}} v(f(t))$. Since v is a sieve for $\mathscr{O}$, this union is not in $\mathscr{O}$, i.e. $F(u) \notin \mathscr{O}$.

We have shown $F^{-1}(\mathscr{O}) = X$. ⊣

The following result shows for which level 2 families $\mathscr{O}$ of Table 2 (cf. Proposition 4.1) one can apply Proposition 6.5.

Proposition 6.6. 1. The family $\mathfrak{P}_{<\omega}(\mathbb{N})$ (line 6a of Table 2) satisfies the computable chain condition, hence is second order many-one hard for $\Sigma^0_2(\mathbf{2}^{<\omega})$.
2. The families of $\mathfrak{P}(\mathbb{N})$, $\mathfrak{P}(\mathbb{N})^2$ or $\mathfrak{P}(\mathbb{N}^2)$ of lines 1b,3,5,6c,7 of Table 2) satisfy the computable co-chain condition, hence are second order many-one hard for $\Pi^0_2(\mathbf{2}^{<\omega})$.
3. Whatever be $A \subseteq \mathbb{N}$, the families $\mathfrak{P}(A)$ and $\{A\}$ (cf. lines 2b,4c,4d of Table 2) admit no sieve and no co-sieve (even non computable ones).
4. Though it has no co-sieve, the Scott Π^0_2 family $\mathscr{O} = \{A\}$ where A is infinite c.e. (cf. line 4c of Table 2) is second order many-one hard for $\Pi^0_2(\mathbf{2}^{<\omega})$.

Proof. 1 and 2. Lines 1b,6a : set $X_i = \{0,\dots,i\}$. Lines 3,5,6c : set $(X_i, Y_i) = (\{0,\dots,i+1\},\{0,\dots,i\})$. Line 7 : set $X_i = \{(j,k) \mid j \leq k \leq i\} \setminus \{(0,i)\}$.

3. *No sieve for* $\mathfrak{P}(A)$ (*line* 2*b*). Sieve condition i insures that $\bigcup_{n\in\mathbb{N}} v(\sigma^\frown n) \subseteq A$. Which implies that all $v(\sigma)$'s are included in A. Therefore $\bigcup_{n\in\mathbb{N}} v(\varphi\restriction n) \subseteq A$ for any $\varphi : \mathbb{N}\to\mathbb{N}$, contradicting sieve condition ii.

No co-sieve for $\mathfrak{P}(A)$. Co-sieve condition i insures that $\bigcup_{k\in\mathbb{N}} v(\langle k\rangle) \not\subseteq A$. Which implies that some $v(\langle k\rangle)$ is not included in A. Therefore $\bigcup_{n\in\mathbb{N}} v(\varphi\restriction n) \not\subseteq A$ for any $\varphi : \mathbb{N}\to\mathbb{N}$ such that $\varphi(0) = k$, contradicting co-sieve condition ii.

No sieve for $\{A\}$ (*lines* 4*cd*). Sieve condition i insures that $\bigcup_{n\in\mathbb{N}} v(\sigma^\frown n) = A$. Which implies that all $v(\sigma)$'s are included in A and $\forall x \in A\ \forall\sigma\ \exists n\ x \in v(\sigma^\frown n)$. This allows the inductive construction of $\varphi : \mathbb{N}\to\mathbb{N}$ such that $\forall x\ (x \in A \Rightarrow x \in v(\varphi\restriction(x+1)))$, which yields $\bigcup_{n\in\mathbb{N}} v(\varphi\restriction n) = A$ and contradicts sieve condition ii.

No co-sieve for $\{A\}$. First, observe that co-sieve condition ii insures that all $v(\sigma)$'s are included in A. Then argue as for $\mathfrak{P}(A)$.

4. Fix a computable enumeration $(a_i)_{i\in\mathbb{N}}$ of A and let $Tot = \{e \mid W_e = \mathbb{N}\}$. Consider the monotone Turing machine which, on input e, outputs a_i if and when i appears in W_e. The associated computable map $G : \mathbb{N} \to \mathfrak{P}(\mathbb{N})$ satisfies $G^{-1}(\{A\}) = Tot$. Now, if $C \subseteq \mathbf{2}^{<\omega}$ is any Π^0_2 set and $f : \mathbf{2}^{<\omega} \to \mathbb{N}$ is a many-one reduction of C to Tot, the map $u \mapsto G(f(u))$ is the wanted second order many-one reduction of C to $\{A\}$. ⊣

Remark 6.7. Since the range of all computable maps $\mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ consists of c.e. sets, if $A \subseteq \mathbb{N}$ is not c.e. then only the empty set is second order many-one reducible to $\{A\}$. This rules out any second order many-one hardness in the case of line 4d of Table 2.

PROBLEM 6.8. When A is Π^0_2 and not Π^0_1, Is $\mathfrak{P}(A)$ second order many-one hard for Π^0_2? (cf. line 2b of Table 2).

6.4. From many-one hardness of index sets to second order many-one hardness. The following very simple theorem allows to get second order many-one hardness from usual many-one hardness of index sets. It is the source of a lot of randomness results.

THEOREM 6.9 (Transfer Theorem). Let $index(\mathscr{O}) = \{e \mid W_e \in \mathscr{O}\} \subseteq \mathbb{N}$ where $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$. The following conditions are equivalent:

1. $index(\mathscr{O})$ is many-one hard for Σ^0_n (resp. Π^0_n, resp. Π^1_1) in the usual sense
2. $\mathscr{O}$ is second order many-one hard for $\Sigma^0_n(\mathbf{2}^{<\omega})$ (resp. Π^0_n, resp. Π^1_1).

PROOF. $1 \Rightarrow 2$. Let $A \subseteq \mathbf{2}^{<\omega}$ be Σ^0_n (resp. Π^0_n, resp. Π^1_1) and let $f : \mathbf{2}^{<\omega} \to \mathbb{N}$ be a computable reduction of A to $index(\mathscr{O})$. Let M be the monotone Turing machine which, on input $u \in \mathbf{2}^{<\omega}$, enumerates $W_{f(u)}$. The associated computable total map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ satisfies $F(u) = W_{f(u)}$, so that, for all $u \in \mathbf{2}^{<\omega}$, $u \in A \Leftrightarrow f(u) \in index(\mathscr{O}) \Leftrightarrow W_{f(u)} \in \mathscr{O} \Leftrightarrow F(u) \in \mathscr{O}$. Thus, F is a second order many-one reduction of A to $\mathscr{O}$.

$2 \Rightarrow 1$. Starting from $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$, let $f : \mathbf{2}^{<\omega} \to \mathbb{N}$ be a computable map such that $F(u) = W_{f(u)}$. Then $u \in A \Leftrightarrow F(u) \in \mathscr{O} \Leftrightarrow W_{f(u)} \in \mathscr{O} \Leftrightarrow f(u) \in index(\mathscr{O})$. ⊣

As an application of known many-one hardness results of index sets (cf. Soare's book [17]) plus the ones from Proposition 4.4, we get the following result.

PROPOSITION 6.10. The Scott Σ^0_n (resp. Π^0_n, resp. Π^1_1) families of lines 6a, 8, 10a to 19 of Table 2 and Table 3 (cf. Propositions 4.1, 4.7) are second order many-one hard for $\Sigma^0_n(\mathbf{2}^{<\omega})$ (resp. Π^0_n, resp. Π^1_1).

§7. Criteria for effective Wadge hardness.

7.1. Effective Wadge hardness at level 1.

PROPOSITION 7.1 (Hardness at level Σ^0_1). Let $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ be Scott Σ^0_1. The following conditions are equivalent:

i. $\mathscr{O}$ is second order many-one hard for $\Sigma^0_1(\mathbf{2}^{<\omega})$,
ii. $\mathscr{O}$ is effectively Wadge hard for $\Sigma^0_1(\mathbf{2}^{\omega})$,
iii. $\mathscr{O} \neq \emptyset$ and $\emptyset \notin \mathscr{O}$.

PROOF. i $\Leftrightarrow$ iii is Proposition 6.1.
ii $\Rightarrow$ iii. Same proof as i $\Rightarrow$ ii of Proposition 6.1 with total $F : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$.
iii $\Rightarrow$ ii. Let $A \neq \emptyset$ be finite such that $\mathscr{O} \supseteq \mathscr{B}_A$. In particular, $A \in \mathscr{O}$. Let $\mathscr{X}$ be $\Sigma^0_1(\mathbf{2}^{\omega})$. There exists a c.e. subset X of $\mathbf{2}^{<\omega}$ such that $\mathscr{X} = X\mathbf{2}^{\omega}$. Define a total computable map $F : \mathbf{2}^{\omega} \to \mathfrak{P}(\mathbb{N})$ such that $F(\alpha) = A$ if α has some prefix in X and $F(\alpha) = \emptyset$ otherwise. Then, $F^{-1}(\mathscr{O}) = \mathscr{X}$. ⊣

Proposition 6.2 has its analog with effective Wadge hardness.

PROPOSITION 7.2. The family $\{X \in \mathfrak{P}(\mathbb{N}) \mid X \supseteq A\}$ (line 1a of Table 2) is effectively Wadge hard for $\Sigma^0_1(\mathbf{2}^{\omega})$ if $A \subseteq \mathbb{N}$ is finite non empty.

The families $\mathfrak{P}(A)$ with $A \neq \mathbb{N}$ Π^0_1 (line 2a), $\{\emptyset\}$ (line 4a) and $\{X \mid |X| \leq p\}$ (line 6b) from Table 2 are effectively Wadge hard for $\Pi^0_1(\mathbf{2}^{\omega})$.

7.2. Effective Wadge hardness at level 2.

PROPOSITION 7.3. A family $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ is effectively Wadge hard for $\Sigma^0_2(\mathbf{2}^\omega)$ if and only if $\mathscr{O}$ admits a computable sieve.

PROOF. $\Leftarrow$. Similar to the proof of Proposition 6.5. $\Rightarrow$. Consider the $\Sigma^0_2(\mathbf{2}^\omega)$ set $\mathbf{2}^{<\omega}0^\omega$ and let $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ be a computable total map such that $F^{-1}(\mathscr{O}) = \mathbf{2}^{<\omega}0^\omega$. For any extension α of u, only u or a prefix of u has been read at step $|u|$. Thus, one can define $\theta : \mathbf{2}^{<\omega} \to \mathfrak{P}_{<\omega}(\mathbb{N})$ so that $\theta(u)$ is the approximation of $F(\alpha)$ obtained at step $|u|$ (i.e., the current output) for any extension α of u. In particular, $F(\alpha) = \bigcup_{t\in\mathbb{N}} \theta(\alpha \restriction t)$. Define $v : \mathbb{N}^{<\omega} \to \mathfrak{P}_{<\omega}(\mathbb{N})^\ell$ as follows: $v(\langle s_0, s_1, \dots, s_k\rangle) = \theta(0^{s_0}10^{s_1}1\dots 0^{s_k})$. Clearly, for $(s_i)_{i\in\mathbb{N}} \in \mathbb{N}^{<\omega}$, $\lim_{n\to\infty} v(\langle s_0, s_1, \dots, s_k, n\rangle) = F(0^{s_0}10^{s_1}1\dots 0^{s_k}10^\omega) \in \mathscr{O}$, and $\lim_{i\to\infty} v(\langle s_0, s_1, \dots, s_i\rangle) = F(0^{s_0}10^{s_1}10^{s_2}\dots) \notin \mathscr{O}$ so that v is a computable sieve for $\mathscr{O}$. ⊣

Using Proposition 6.6, the above characterization Proposition 7.3 allows to get positive and negative effective Wadge hardness results.

PROPOSITION 7.4. 1. The family $\mathfrak{P}_{<\omega}(\mathbb{N})$ (line 6a of Table 2) is effectively Wadge hard for $\Sigma^0_2(\mathbf{2}^\omega)$.
2. The families of lines 1b,3,5,6c,7 of Table 2 are effectively Wadge hard for $\Pi^0_2(\mathbf{2}^\omega)$.
3. Whatever be $A \subseteq \mathbb{N}$, the families $\mathfrak{P}(A)$ and $\{A\}$ (cf. lines 2b,4c,4d of Table 2) are not effectively Wadge hard neither for $\Sigma^0_2(\mathbf{2}^\omega)$ nor for $\Pi^0_2(\mathbf{2}^\omega)$.

PROBLEM 7.5. The almost everywhere or measure hardness of $\mathfrak{P}(A)$ for A non Π^0_1, and that of $\{A\}$ for $A \neq \emptyset$ are open questions.

7.3. Effective (almost everywhere) Wadge hardness at level 3.

THEOREM 7.6. Suppose $\mathscr{O} \subset \mathfrak{P}(\mathbb{N})$ is countable, and contains all cofinite subsets of $\mathbb{N}$. Then,

i. $\mathscr{O}$ is effectively almost everywhere Wadge hard for $\Sigma^0_3(\mathbf{2}^\omega)$.
ii. If $\forall X \in \mathscr{O}\ \emptyset' \not\leq_{\text{Turing}} X$, i.e. no set in $\mathscr{O}$ allows to compute $\emptyset'$ (for instance if all sets in $\mathscr{O}$ are computable), then $\mathscr{O}$ is effectively Wadge hard for $\Sigma^0_3(\mathbf{2}^\omega)$.

PROOF. *Main idea of the proof.* We develop a variation of Rogers' classical proof of the Σ^0_3 completeness of the index set of the class of computable sets. Let $R \subseteq \mathbb{N}^3$ be computable such that, for all $\alpha \in \mathbf{2}^\omega$,

$$\alpha \in \mathscr{X} \Leftrightarrow \exists x\ \exists^\infty y\ R(x, \alpha \restriction y).$$

Observe that $\exists^\infty y\ R(x, \alpha \restriction y) \Leftrightarrow \exists^\infty y \geq x\ R(x, \alpha \restriction y)$. We define a computable map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ with the following properties. For all $\alpha \in \mathbf{2}^\omega$,

$$\alpha \in \mathscr{X} \Rightarrow F(\alpha) \text{ is cofinite,} \tag{1}$$

$$\alpha \notin \mathscr{X} \Rightarrow \alpha' \text{ is Turing computable in } F(\alpha). \tag{2}$$

Proof of point i. Let $\mathscr{E} \subseteq \mathbf{2}^\omega$ be the countable family of $\alpha \in \mathbf{2}^\omega$ such that the jump α' is Turing computable in some set in $\mathscr{O}$. From (2), we get

$$\alpha \notin \mathscr{X} \Rightarrow F(\alpha) \notin \mathscr{O}\ \vee\ \alpha \in \mathscr{E}. \tag{3}$$

Since cofinite sets are in $\mathscr{O}$ and $\mathscr{E}$ is countable (hence of measure zero), conditions (1) and (3) insure that $\mathscr{X} =_{a.e.} F^{-1}(\mathscr{O})$.

Proof of point ii. Since $\alpha' \leq_T F(\alpha)$ implies $\emptyset' \leq_T F(\alpha)$, the extra hypothesis of point ii of the Theorem insures that $\alpha \notin \mathscr{X} \Rightarrow F(\alpha) \notin \mathscr{O}$. Whence $\mathscr{X} = F^{-1}(\mathscr{O})$.

Construction of F. We define F such that

- If x_0 is least such that $\exists^\infty y\ R(x_0, \alpha \restriction y)$ then $F(\alpha)$ is cofinite
- If $\forall x\ \neg(\exists^\infty y\ R(x, \alpha \restriction y))$ then $F(\alpha) = \bigcup_{x \in \mathbb{N}} [a_x, b_x]$ where $0 = a_0 \leq b_0 < a_1 \leq b_1 < a_2 \leq b_2 < \ldots$, and for all x, $a_{x+1} - b_x = 2 - \alpha'(x)$

To get F computable, we define a computable monotone increasing $\theta : \mathbf{2}^{<\omega} \to \mathfrak{P}_{<\omega}(\mathbb{N})$ such that $F(\alpha) = \bigcup_{n \in \mathbb{N}} \theta(\alpha \restriction n)$ for all $\alpha \in \mathbf{2}^\omega$. Consider a machine $\mathscr{M}$ which enumerates α' using oracle α and let α'_t be the set of $n \leq t$ which are enumerated by $\mathscr{M}$ within t steps using only questions to the oracle about $\alpha \restriction t$. The function θ is such that

$$\theta(u) = \bigcup_{x \leq |u|} [a^u_x, b^u_x] \text{ where } \begin{cases} a^u_0 \leq b^u_0 < a_1 \leq b^u_1 < \cdots < a^u_{|u|} \leq b^u_{|u|}, \\ \forall x \leq |u| \quad b^u_x - a^u_x \equiv \alpha'_{|u|}(x) \mod 2, \\ \forall x < |u| \quad a^u_{x+1} - b^u_x \geq 2 - \alpha'_{|u|}(x) \end{cases}$$

and, letting $i \in \{0, 1\}$, θ is defined by induction on $|u|$ as follows:

- $\theta(\varepsilon) = [a^\varepsilon_0, b^\varepsilon_0] = [0, \alpha'_0(0)]$,
- If $\forall x \leq |u|\ \neg R(x, ui)$ then, for all $x \leq |u|$ we let $a^{ui}_x = a^u_x$ and $b^{ui}_x = b^u_x + (\alpha'_{|ui|}(x) - \alpha'_{|u|}(x))$ (i.e., if and when x appears in α', we increment the length of the interval $[a^{ui}_x, b^{ui}_x]$ so that it becomes odd).
  We also set $a^{ui}_{|ui|} = b^{ui}_{|u|} + 2$ and $b^{ui}_{|ui|} = a^{ui}_{|ui|} + (2 - \alpha'_{|ui|}(|ui|))$.
- If $\exists x \leq |u|\ R(x, ui)$ and ξ is least such, then we let
  - $a^{ui}_x = a^u_x$ and $b^{ui}_x = b^u_x + (\alpha'_{|ui|}(x) - \alpha'_{|u|}(x))$ for all $x <_{\text{pref}} \xi$.
  - $a^{ui}_\xi = a^u_\xi$ and b^{ui}_ξ is equal to $b^u_{|u|}$ or $b^u_{|u|} + 1$ so that $b^{ui}_\xi - a^{ui}_\xi \equiv \alpha'_{|ui|}(\xi)$ mod 2 (i.e., $[a^{ui}_\xi, b^{ui}_\xi]$ covers all intervals $[a^u_y, b^u_y]$ for $y = \xi, \ldots, |u|$).
  - For $z = \xi + 1, \ldots, |ui|$, let $a^{ui}_z = b^{ui}_{z-1} + 3$ and $b^{ui}_z = a^{ui}_z + \alpha'_{|ui|}(z)$.

We now prove (1). Suppose $\alpha \in \mathscr{X}$ and let x_0 be least such that $\exists^\infty y\ R(x_0, \alpha \restriction y)$. Let $N = \max\{y \mid \exists x < x_0\ R(x, \alpha \restriction y)\}$. An easy induction on $x < x_0$ shows that, for $x < x_0$, the intervals $[a^{\alpha\restriction y}_x, b^{\alpha\restriction y}_x]$ are constant for $y \geq N$. On the opposite, $[a^{\alpha\restriction y}_{x_0}, b^{\alpha\restriction y}_{x_0}]$ tends to $[a^{\alpha\restriction N}_{x_0}, +\infty)$ when y tends to $+\infty$. Thus, $F(\alpha)$ is cofinite.

We finally prove (2). Suppose $\alpha \notin \mathscr{X}$. For $x \in \mathbb{N}$, let $Y(x) = \max\{y \mid \exists x' \leq x\ R(x', \alpha \restriction y)\}$. An easy induction shows that, for every x, the left endpoint $a^{\alpha\restriction y}_x$ remains constant for $y \geq Y(x)$ and the right endpoint $b^{\alpha\restriction y}_x$ is incremented at most once if and when x appears in α'. Let a_x, b_x be the limit values. Then $b_x - a_x \equiv \alpha'_{|u|}(x) \mod 2$, i.e. $\alpha'(x)$ is the parity of the length of the $x+1$-th interval in $F(\alpha)$. Thus, α' is computable with oracle $F(\alpha)$. $\dashv$

As an application of Theorem 7.6, we get

THEOREM 7.7. 1. The families of cofinite sets $\subseteq \mathbb{N}$ (line 8 of Table 2) and of computable sets (line 10a of Table 3) are effectively Wadge hard for $\Sigma^0_3(\mathbf{2}^\omega)$.

2. Whatever be $A \subseteq \mathbb{N}$, the families of A-c.e. sets and of boolean combinations of A-c.e. sets (lines 9a,9b,9c of Table 3) are effectively almost everywhere Wadge hard for $\Sigma^0_3(\mathbf{2}^\omega)$ but not effectively Wadge hard.
3. The families of co-c.e. sets and of c.e. non simple sets (lines 10b,11b of Table 3) are effectively almost everywhere Wadge hard for $\Sigma^0_3(\mathbf{2}^\omega)$.

PROOF. The positive assertions are straightforward applications of Theorem 7.6. For the negative assertion in Point 2, recall that effective Wadge hardness implies second order many-one hard (Corollary 5.15). Now, since the families $\mathscr{O}$ of lines 9a,9b,9c contain all c.e. sets, we have $F^{-1}(\mathscr{O}) = \mathbf{2}^{<\omega}$ for every computable map $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$, which rules out any second order many-one hardness. ⊣

PROBLEM 7.8. Effective Wadge hardness of the family of co-c.e. sets and of the family of c.e. non simple sets are open questions.

REMARK 7.9. The condition *"$\mathscr{O}$ is countable"* in Theorem 7.6 cannot be replaced by $\mu(\mathscr{O}) = 0$. In fact, this last hypothesis does not imply that $\mu(\mathscr{E}) = \mu(\{\alpha \mid \exists\beta \in \mathscr{O}\ \alpha' \leq_T \beta\}) = 0$. Consider, for instance $\mathscr{O} = \{\alpha \mid \forall n\ \alpha(2n) = 0\}$.

7.4. Effective almost everywhere Wadge hardness at level 4. We review the results leading to the $\Sigma_3^{0,C}$ many-one completeness of the index set $\{e \mid C \leq_T W_e\}$ where C is c.e. and not computable (which is Σ_3^0 or Σ_4^0 completeness depending whether C is low or high) and state their counterparts when an argument in $\mathbf{2}^\omega$ is added or replaces an argument in $\mathbf{2}^{<\omega}$. The final result is Theorem 7.18 which is the Wadge hardness counterpart to this completeness result. The proof uses a second order version of Yates Index set theorem, that we prove in the next subsections. We follow notations from Soare's book as much as possible.

7.4.1. *Strong thickness lemma with a second order argument.*

LEMMA 7.10 (Strong thickness lemma, [17] p.135). Suppose $C \subset \mathbb{N}$ is c.e. and not computable and $B \subseteq \mathbb{N}^2$ is c.e. Then there exists a c.e. $A \subseteq B$ such that, letting $B_e = \{x \mid (x, e) \in B\}$ and $B^{[<e]} = \bigcup_{j<e} B_j$,

1. $A \leq_T B$,
2. (a) $\forall e\ (C \not\leq_T B^{[<e]} \Rightarrow B_e \setminus A_e$ is finite),
   (b) $(\forall e\ C \not\leq_T B^{[<e]}) \Rightarrow C \not\leq_T A$.

A second order counterpart is as follows, where $\alpha \oplus X$ is any computable coding of the pair $(\alpha, X) \in \mathbf{2}^\omega \times \mathfrak{P}(\mathbb{N}^d)$ as a set, for instance

$$\alpha \oplus X = 2\alpha^{-1}(1) \times \mathbb{N}^{d-1} \cup \{(2x+1, \vec{y}) \mid (x, \vec{y}) \in X\}.$$

LEMMA 7.11 (Second order strong thickness lemma). Suppose $C \subset \mathbb{N}$ is c.e. and $\mathscr{B} \subseteq \mathbf{2}^\omega \times \mathbb{N}^2$ is $\Sigma_1^0(\mathbf{2}^\omega \times \mathbb{N}^2)$. Then there exists a $\Sigma_1^0(\mathbf{2}^\omega \times \mathbb{N}^2)$ subset $\mathscr{A} \subseteq \mathscr{B}$ such that, for all α,

1. $\mathscr{A}_\alpha \leq_T \alpha \oplus \mathscr{B}_\alpha$,
2. if $C \not\leq_T \alpha$ then
   (a) $\forall e\ (C \not\leq_T \alpha \oplus \mathscr{B}^{[\alpha,<e]} \Rightarrow \mathscr{B}_{\alpha,e} \setminus \mathscr{A}_{\alpha,e}$ is finite),
   (b) $(\forall e\ C \not\leq_T \alpha \oplus \mathscr{B}^{[\alpha,<e]}) \Rightarrow C \not\leq_T \alpha \oplus \mathscr{A}_\alpha$.

7.4.2. *Yates Representation theorem with a second order argument.* We restate Lemma XII 1.4 and Theorem XII 1.3 from Soare's book [17] (pp.242–243).

LEMMA 7.12. If $C \subseteq \mathbb{N}$ is a c.e. set and $R \subseteq \mathbb{N}$ is $\Pi_2^{0,C}$ then there is a c.e. set $B \subseteq \mathbb{N}^2$ which is C-computable and such that, for all $e \in \mathbb{N}$,

$$e \in R \Rightarrow B_e \equiv_T C,$$
$$e \notin R \Rightarrow B_e \text{ is computable.}$$

THEOREM 7.13 (Yates Representation theorem). Let $C \subseteq \mathbb{N}$ be any c.e. set. For any $\Sigma_3^{0,C}$ set $S \subseteq \mathbb{N}$ there is a c.e. set $B \subseteq \mathbb{N}^3$ which is C-computable and such that, for all $k \in \mathbb{N}$,

$$k \in S \Rightarrow \exists e_0\ [(\forall e \geq e_0\ B_{k,e} \equiv_T C)\ \wedge\ (\forall e < e_0\ B_{k,e} \text{ is computable})],$$
$$k \notin S \Rightarrow \forall e\ B_{k,e} \text{ is computable.}$$

These results can be reformulated as follows in a second order context. The proofs are slight modifications of those in [17].

LEMMA 7.14. Let $C \subseteq \mathbb{N}$ be any c.e. set. To any $\Pi_2^{0,C}(\mathbf{2}^\omega)$ set $\mathscr{R} \subseteq \mathbf{2}^\omega \times \mathbb{N}$ one can associate a set $\mathscr{B} \subseteq \mathbf{2}^\omega \times \mathbb{N}^2$ which is $\Sigma_1^0(\mathbf{2}^\omega \times \mathbb{N}^2)$ and C-computable and such that, for all α and e,

$$(\alpha, e) \in \mathscr{R} \Rightarrow \mathscr{B}_{\alpha,e} \equiv_T \alpha \oplus C,$$
$$(\alpha, e) \notin \mathscr{R} \Rightarrow \mathscr{B}_{\alpha,e} \text{ is computable.}$$

PROOF. To simplify notations, we shall get $\mathscr{B} \subseteq \mathbf{2}^\omega \times \mathbb{N}^3$. A simple computable coding of the two last components makes $\mathscr{B}$ included in $\mathbf{2}^\omega \times \mathbb{N}^2$. Let $\rho \subseteq \mathbf{2}^{<\omega} \times \mathbf{2}^{<\omega} \times \mathbb{N}^3$ be computable such that $(\alpha, e) \in \mathscr{R} \Leftrightarrow \forall y\ \exists z\ (\alpha \upharpoonright z, C \upharpoonright z, e, y, z) \in \rho$. Denote by C_t the finite approximation of C obtained at step t of some computable enumeration of C. Define integers $n^t_{\alpha,e,y}$ by induction on t as follows: $n^0_{\alpha,e,y} = 0$ and

$$n^{t+1}_{\alpha,e,y} = \begin{cases} n^t_{\alpha,e,y} & \text{if } \forall y' \leq y\ \exists z \leq t+1\ (\alpha \upharpoonright z, C_{t+1} \upharpoonright z, e, y, z) \in \rho, \\ n^t_{\alpha,e,y} + 1 & \text{otherwise.} \end{cases}$$

Let $D_y = 3\mathbb{N} + 2$ if $y \in C$ and $D_y = \emptyset$ if $y \notin C$. Define $\mathscr{B} \subseteq \mathbf{2}^\omega \times \mathbb{N}^3$ as follows

$$\mathscr{B}_{\alpha,e,y} = (3\mathbb{N} + \alpha(y))\ \cup\ D_y\ \cup\ \bigcup_{t \in \mathbb{N}} [0, n^t_{\alpha,e,y}].$$

The construction of $\mathscr{B}$ insures that it is Σ_1^0 and computable in C.

Suppose $(\alpha, e) \in \mathscr{R}$. Then, for all y the sequence $(n^t_{\alpha,e,y})_t$ is eventually constant, so that $\mathscr{B}_{\alpha,e,y} = (3\mathbb{N} + \alpha(y)) \cup D_y\ \cup [0, n]$ for some n. Observe that $y \notin C$ if and only if $\exists x\ 3x + 2 \notin \mathscr{B}_{\alpha,e,y}$, which proves that C is co-c.e. in $\mathscr{B}_{\alpha,e}$, hence computable from $\mathscr{B}_{\alpha,e}$ (since C is c.e.). Similarly, for $\varepsilon = 0, 1$, $\alpha(y) = \varepsilon$ if and only if $\exists x\ 3x + (1 - \varepsilon) \notin \mathscr{B}_{\alpha,e,y}$, which proves that α is computable from $\mathscr{B}_{\alpha,e}$. Thus, $\alpha \oplus C \leq_T \mathscr{B}_{\alpha,e}$. Finally, $\mathscr{B}_{\alpha,e} \leq_T \alpha \oplus C$ since $\mathscr{B}$ is computable in C.

Suppose $(\alpha, e) \notin \mathscr{R}$. Let y_0 be least such that $\forall z\ (\alpha \upharpoonright z, C \upharpoonright z, e, y_0, z) \notin \rho$. Then $\mathscr{B}_{\alpha,e,y} = \mathbb{N}$ for all $y \geq y_0$ and $\mathscr{B}_{\alpha,e,y}$ is of the form $(3\mathbb{N} + \alpha(y)) \cup\ D_y \cup [0, n_y]$ for all $y < y_0$ where D_y is $3\mathbb{N} + 2$ or $\emptyset$. Clearly, $\mathscr{B}_{\alpha,e}$ is then computable. ⊣

THEOREM 7.15 (Second order Yates Representation theorem). Let $C \subseteq \mathbb{N}$ be any c.e. set. To any $\Sigma_3^{0,C}(\mathbf{2}^\omega)$ set $\mathcal{S} \subseteq \mathbf{2}^\omega$ one can associate a set $\mathscr{B} \subseteq \mathbf{2}^\omega \times \mathbb{N}^2$ which is $\Sigma_1^0(\mathbf{2}^\omega \times \mathbb{N}^2)$, C-computable and such that, for all α,

$$\alpha \in \mathcal{S} \Rightarrow \exists e_0\ [(\forall e \geq e_0\ \mathscr{B}_{\alpha,e} \equiv_T \alpha \oplus C)\ \wedge\ (\forall e < e_0\ \mathscr{B}_{\alpha,e} \text{ is computable})],$$
$$\alpha \notin \mathcal{S} \Rightarrow \forall e\ \mathscr{B}_{\alpha,e} \text{ is computable.}$$

Proof. Let $\mathscr{R} \subseteq \mathbf{2}^\omega \times \mathbb{N}$ be $\Pi_2^{0,C}(\mathbf{2}^\omega \times \mathbb{N})$ such that $\alpha \in \mathscr{S} \Leftrightarrow \exists e\ \mathscr{R}(\alpha, e)$. Replacing $\mathscr{R}(\alpha, e)$ by $\exists e' \leq e\ \mathscr{R}(\alpha, e')$, we have

$$\alpha \in \mathscr{S} \Rightarrow \exists e_0\ [(\forall e \geq e_0\ (\alpha, e) \in \mathscr{R}) \wedge (\forall e < e_0\ (\alpha, e) \notin \mathscr{R})],$$
$$\alpha \notin \mathscr{S} \Rightarrow \forall e\ (\alpha, e) \notin \mathscr{R}.$$

To conclude, apply Lemma 7.14 above. ⊣

7.4.3. *Yates index set with a second order argument.*

Theorem 7.16 (Yates index set theorem). Given c.e. sets $C, D, S \subset \mathbb{N}$ such that $D <_T C$ and S is $\Sigma_3^{0,C}$, there is a computable function g such that

$$\forall k\ \ D \leq_T W_{g(k)} \leq_T C \qquad \text{and} \qquad \forall k\ (k \in S \Leftrightarrow C \equiv_T W_{g(k)}).$$

In particular, $\{e \in \mathbb{N} \mid W_e \equiv_T C\}$ and $\{e \in \mathbb{N} \mid C \leq_T W_e\}$ are $\Sigma_3^{0,C}$complete.

A second order analogue is as follows.

Theorem 7.17 (Second order Yates index set theorem). Let $C \subset \mathbb{N}$ be c.e. and not computable. For every $\mathscr{S} \subseteq \mathbf{2}^\omega$ which is $\Sigma_3^{0,C}(\mathbf{2}^\omega)$, there exists a computable map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N}^2)$ such that, for all α,

$$F(\alpha) \leq_T \alpha \oplus C,\ \ \alpha \in S \Rightarrow \alpha \oplus C \leq_T F(\alpha),\ \ C \not\leq_T \alpha \Rightarrow (\alpha \notin S \Rightarrow C \not\leq_T F(\alpha)).$$

Proof. For point 1, we argue as in [17]. Let $\mathscr{B} \subseteq \mathbf{2}^\omega \times \mathbb{N}^2$ be given by Theorem 7.15. Apply the second order Thickness Lemma 7.11 to get $\mathscr{A} \subseteq \mathscr{B}$ which is again $\Sigma_1^0(\mathbf{2}^\omega \times \mathbb{N}^2)$ and such that, for all α,

$$\mathscr{A}_\alpha \leq_T \alpha \oplus \mathscr{B}_\alpha, \tag{4}$$

$$C \not\leq_T \alpha \Rightarrow \begin{cases} \forall e\ (C \not\leq_T \alpha \oplus \mathscr{B}^{[\alpha,<e]} \Rightarrow B_{\alpha,e} \setminus \mathscr{A}_{\alpha,e} \text{ is finite}), \\ (\forall e\ C \not\leq_T \alpha \oplus \mathscr{B}^{[\alpha,<e]}) \Rightarrow C \not\leq_T \alpha \oplus \mathscr{A}_\alpha. \end{cases} \tag{5}$$

Since $\mathscr{A}$ is $\Sigma_1^0(\mathbf{2}^\omega \times \mathbb{N}^2)$, the map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N}^2)$ such that $F(\alpha) = \mathscr{A}_\alpha$ is computable (in the sense of Definition 2.1).

Since $\mathscr{B} \leq_T C$ we have $\mathscr{B}_\alpha \leq_T \alpha \oplus C$. Using (4) we get $F(\alpha) = \mathscr{A}_\alpha \leq_T \alpha \oplus C$.

Suppose $\alpha \in \mathscr{S}$. Then $\mathscr{B}_{\alpha,e} \equiv_T \alpha \oplus C$ for e big enough. Let e be least such that $\mathscr{B}_{\alpha,e} \equiv_T \alpha \oplus C$. Then $\mathscr{B}_{\alpha,j}$ is computable for all $j < e$ so that $\mathscr{B}^{[\alpha,<e]}$ is computable. Applying the first line of (5), we see that $\mathscr{A}_{\alpha,e}$ and $\mathscr{B}_{\alpha,e}$ differ only on finitely many elements. In particular, $\mathscr{A}_{\alpha,e} \equiv \alpha \oplus C$. Thus, $\alpha \oplus C \leq_T \mathscr{A}_{\alpha,e} \leq_T \mathscr{A}_\alpha = F(\alpha)$.

Suppose $\alpha \notin \mathscr{S}$ and $C \not\leq_T \alpha$. Then $\mathscr{B}_{\alpha,e}$ and $\mathscr{B}^{[\alpha,<e]}$ are computable for all e. Applying the second line of (5), we see that $C \not\leq_T \alpha \oplus \mathscr{A}_\alpha$ hence $C \not\leq_T \mathscr{A}_\alpha = F(\alpha)$. ⊣

Theorem 7.18.

1. The family $\{X \in \mathfrak{P}(\mathbb{N}) \mid C \leq_T X\}$ is effectively almost everywhere Wadge hard for $\Sigma_3^{0,C}(\mathbf{2}^\omega)$ whenever $C \subset \mathbb{N}$ is a non computable c.e. set. In particular, this is effective almost everywhere hardness for $\Sigma_3^0(\mathbf{2}^\omega)$ if C is low c.e. for $\Sigma_4^0(\mathbf{2}^\omega)$ if C is high c.e. (lines 14a, 15a of Table 3).
2. The family $\{(X, Y) \in \mathfrak{P}(\mathbb{N})^2 \mid X \leq_T Y\}$ is effectively almost everywhere Wadge hard for $\Sigma_4^0(\mathbf{2}^\omega)$ (line 16a of Table 3).

Proof. 1. Let $\mathscr{S}$ and F be as in Theorem 7.17. On the set $\{\alpha \mid C \not\leq_T \alpha\}$ we have $\alpha \in \mathscr{S} \Leftrightarrow C \leq_T F(\alpha)$. Since C is not computable, Sacks' theorem ([14] p.154) insures that $\{\alpha \mid C \not\leq_T \alpha\}$ has measure 1. Thus, F is an effective almost everywhere Wadge reduction of $\mathscr{S}$ to $\mathscr{O}$.

2. Let $\mathcal{S} \subseteq \mathbf{2}^\omega$ be $\Sigma^0_4(\mathbf{2}^\omega)$ and let F be as above for $\mathcal{S}$ and $C = \emptyset'$ (which is high). The map $G : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})^2$ such that $G(\alpha) = (\emptyset', F(\alpha))$ is then an effective almost everywhere Wadge reduction of $\mathcal{S}$ to $\{(X, Y) \in \mathfrak{P}(\mathbb{N})^2 \mid X \leq_T Y\}$. ⊣

Problem 7.19. In the classical many-one framework, Yates index set Theorem also insures the many-one hardness of the index set $\{e \in \mathbb{N} \mid W_e \equiv_T C\}$. However, the above argument fails to give effective Wadge hardness of the set $\{X \in \mathfrak{P}(\mathbb{N}) \mid X \equiv_T C\}$. In the proof we have (for $\alpha \in \mathcal{S}$) $F(\alpha) \equiv_T \alpha \oplus C$ instead of $F(\alpha) \equiv_T C$.

How much effective Wadge hard is this set $\mathcal{O}$ is an open problem. It cannot be $\Sigma^0_4(\mathbf{2}^\omega)$ hard. In fact, every singleton family in $\mathfrak{P}(\mathbb{N})$ is Scott $\mathbf{\Pi}^0_2$ (the boldface classes are the Scott Borel classes of finite levels, which are defined by forgetting the condition $\mathcal{C}$ is c.e. in Definition 3.1). So that, being countable, $\mathcal{O}$ is Scott $\mathbf{\Sigma}^0_3$. And $F^{-1}(\mathcal{O})$ is $\mathbf{\Sigma}^0_3(\mathbf{2}^\omega)$ whereas there exists some $\Sigma^0_4(\mathbf{2}^\omega)$ family which is not $\mathbf{\Sigma}^0_3(\mathbf{2}^\omega)$.

7.5. From many-one hardness of index sets to effective Wadge hardness. As we experienced in §7.4, some many-one hardness results for index sets of c.e. sets relativize – with easy changes in the proof – to index sets of A-c.e. sets uniformly in the oracle $\alpha \in \mathbf{2}^\omega$. This leads to effective Wadge hardness results. Theorem 7.20 is a straightforward formalization of this observation.

Theorem 7.20 (Transfer Theorem). Suppose $\mathcal{O} \subseteq \mathfrak{P}(\mathbb{N})$ is such that

i. *index*$(\mathcal{O}) = \{e \mid W_e \in \mathcal{O}\}$ is many-one hard for $\Sigma^0_n(\mathbb{N})$,
ii. this hardness relativizes uniformly, i.e., for every Σ^0_n formula $\psi(x, f)$ of the language of second order arithmetics with free variables x and f varying in $\mathbb{N}$ and $\mathbf{2}^\omega$, there exists some total computable $\theta : \mathbb{N} \to \mathbb{N}$ (depending on ψ only) such that

$$\forall \alpha\, \forall i\; (\psi(i, \alpha) \;\Leftrightarrow\; W^\alpha_{\theta(i)} \in \mathcal{O}). \tag{6}$$

Then $\mathcal{O}$ is effectively Wadge hard for the class of $\Sigma^0_n(\mathbf{2}^\omega)$ sets.

Idem with Π^0_n or Π^1_1 in place of Σ^0_n.

Proof. Let $\mathcal{A}$ be $\Sigma^0_n(\mathbf{2}^\omega)$. Apply the hypothesis to $\mathcal{A} \times \mathbb{N}$ to get a computable map $\theta : \mathbb{N} \to \mathbb{N}$ satisfying (6). The computable map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ such that $F(\alpha) = W^\alpha_{\theta(0)}$ is an effective Wadge reduction of $\mathcal{A}$ to $\mathcal{O}$. ⊣

As an application of Theorem 7.20, we get a new proof of level 1 and 2 effective Wadge hardness for the $\mathcal{O}$'s of lines 6bc, 7 of Table 2, and also

Proposition 7.21. 1. Let $w = w_1 ... w_{i+j}$ be a non empty word in the alphabet $\{FIN, COF\}$ containing i occurrences of FIN and j occurrences of COF. Then $Set(w)$ (cf. Definition 4.2 and lines 6a,8, 18 of Table 2) is effectively Wadge hard for $\Sigma^0_{i+2j+1}(\mathbf{2}^\omega)$.

2. The family $\{X \in \mathfrak{P}(\mathbb{N}^2) \mid X \text{ is well-founded}\}$ (line 19 of Table 2) is effectively Wadge hard for $\Pi^1_1(\mathbf{2}^\omega)$.

Proof. Observe that the proof of Proposition 4.4 (and that for the index set of well-founded c.e. subsets of $\mathbb{N}^2$) relativizes in such a way that the conditions of Theorem 7.20 are satisfied. ⊣

§8. Criteria for special hardness.

8.1. Open special hardness at levels 1 and 2. It happens that open special hardness for $\Sigma^0_1(\mathbf{2}^{<\omega})$ and $\Sigma^0_2(\mathbf{2}^{<\omega})$ coincide and are always true except in a trivial case.

PROPOSITION 8.1. *Let $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$. The following conditions are equivalent:*

i. *$\mathscr{O}$ is open special hard for $\Sigma^0_1(\mathbf{2}^{<\omega})$,*
ii. *$\mathscr{O}$ is open special hard for $\Sigma^0_2(\mathbf{2}^{<\omega})$,*
iii. *$\mathscr{O}$ contains some c.e. set.*

PROOF. ii $\Rightarrow$ i and i $\Rightarrow$ iii are obvious. It remains to prove iii $\Rightarrow$ ii. Suppose iii and let $X \in \mathscr{O}$ be c.e. In case $X \neq \emptyset$, let $f : \mathbb{N} \to \mathbb{N}$ be a computable enumeration of X. Let $A \subseteq \mathbf{2}^{<\omega}$ be Σ^0_2. Applying Lemma 5.10, let $B \subseteq \mathbf{2}^{<\omega}$ be prefix-free and Π^0_1 such that $A\mathbf{2}^\omega = B\mathbf{2}^\omega$. Let $B = \{u \mid \forall y\ R(y,u)\}$ where R is computable. Let $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ be the self-delimited constant partial map with value X and domain B. Clearly, $F^{-1}(\mathscr{O}) = B$ so that $F^{-1}(\mathscr{O})\mathbf{2}^\omega = A\mathbf{2}^\omega$. We show that F is partial computable in the sense of Definition 2.2. Consider the monotone Turing machine M which behaves as follows. At time t, whatever be the input, the current output of M is $\{f(s) \mid s < t\}$ if $X \neq \emptyset$. Otherwise it is empty. The computation of M consists of successive phases corresponding to the successive prefixes of the input read by the input head: during phase i, M has read u_i with length i. The role of phase i is to test condition $\forall y\ R(y,u_i)$, i.e., to test whether u_i is in B. Let t_i be the starting time of phase i. If the condition is true then phase i lasts forever. Else phase i halts at time $t_i + z$ where z is the least y such that $R(y,u_i)$ fails. It is clear that M is self-delimited with domain B and computes F. ⊣

8.2. No open special hardness at level ≥ 3. Special hardness and open special hardness happen to be void concepts at levels ≥ 3. First, recall a well-known result.

PROPOSITION 8.2. *Let $X \subseteq \mathbf{2}^{<\omega}$ and $u \in \mathbf{2}^{<\omega}$. Then $u\mathbf{2}^\omega \subseteq X\mathbf{2}^\omega$ if and only if there exists $n \geq |u|$ such that every extension of u of length n extends some word in X.*

PROOF. Consider the tree T of words which extend u and have no prefix in X. Since $u\mathbf{2}^\omega \subseteq X\mathbf{2}^\omega$, this tree has no infinite branch. By König's lemma, T has to be finite. To conclude, let n be greater than the height of T. ⊣

PROPOSITION 8.3. *There exists a Δ^0_3 set A which is not open special reducible to any family $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$. In particular, there is no open special hard family $\mathscr{O}$ for any class $\mathscr{C} \subseteq \mathfrak{P}(\mathbf{2}^{<\omega})$ which includes Δ^0_3.*

PROOF. Choose $\xi \in \mathbf{2}^\omega$ such that $\xi^{-1}(1)$ is Δ^0_3 and not $\Sigma^0_2 \wedge \Pi^0_2$ and let $A = \{u\varepsilon \mid u \leq_{\text{pref}} \xi \wedge \varepsilon = 1 - \xi(|u|)\}$. Clearly, $A\mathbf{2}^\omega = \mathbf{2}^\omega \setminus \{\xi\}$. To prove the Proposition, it suffices to show that if $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ is self-delimited computable and $X \subseteq dom(F)$ then $A\mathbf{2}^\omega \neq X\mathbf{2}^\omega$. Arguing by way of contradiction, suppose $A\mathbf{2}^\omega = X\mathbf{2}^\omega$. Then $\mathbf{2}^\omega \setminus \{\xi\} = A\mathbf{2}^\omega = X\mathbf{2}^\omega \subseteq dom(F)\mathbf{2}^\omega$. There are only two cases for this last inclusion.

Case $X\mathbf{2}^\omega \subsetneq dom(F)\mathbf{2}^\omega$. Since $A\mathbf{2}^\omega$ is the complement of a singleton set, we necessarily have $dom(F)\mathbf{2}^\omega = \mathbf{2}^\omega$. Let n be as in Proposition 8.2 applied with u being the empty word. Any length n word has a prefix in $dom(F)$. Since $dom(F)$ is prefix-free, this ensures that $dom(F)$ hence also X is finite. But then $X\mathbf{2}^\omega$ is open and closed so that it cannot be equal to $A\mathbf{2}^\omega = \mathbf{2}^\omega \setminus \{\xi\}$ which is not closed.

Case $X\mathbf{2}^\omega = dom(F)\mathbf{2}^\omega$. Then $\mathbf{2}^\omega \setminus \{\xi\} = A\mathbf{2}^\omega = dom(F)\mathbf{2}^\omega$. Consider the map $\rho : \mathfrak{P}(\mathbf{2}^{<\omega}) \to \mathfrak{P}(\mathbf{2}^{<\omega})$ such that $\rho(Z)$ is the set of minimal words u (relative to the prefix ordering) such that $u\mathbf{2}^\omega \subseteq Z\mathbf{2}^\omega$. Observe that $\rho(A) = A$. Since $A\mathbf{2}^\omega = dom(F)\mathbf{2}^\omega$ and $\rho(Z)$ depends only on $Z\mathbf{2}^\omega$, we have $\rho(domF) = \rho(A) = A$. Now, using Proposition 8.2, $\rho(dom(F))$ can be defined as follows:

$$\begin{aligned} u \in \rho(X) \Leftrightarrow\ & (\exists n\ \forall v \in \mathbf{2}^n\ (u \leq_{\text{pref}} v \Rightarrow \exists x \leq_{\text{pref}} v\ x \in dom(F)) \\ & \wedge (\forall w <_{\text{pref}} u\ \forall n\ \exists v \in \mathbf{2}^n\ (w \leq_{\text{pref}} v \wedge \forall x \leq_{\text{pref}} v\ x \notin dom(F))). \end{aligned}$$

Since $dom(F)$ is $\Sigma^0_1 \wedge \Pi^0_1$ (cf. Proposition 2.4), we see that $\rho(dom(F))$ is $\Sigma^0_2 \wedge \Pi^0_2$. Now, $dom(F) = A$ and we have chosen A not $\Sigma^0_2 \wedge \Pi^0_2$. Contradiction. ⊣

8.3. Almost everywhere and measure special hardness: from Π^0_n to Σ^0_{n+1}. The analog of Theorem 5.11 holds for the special framework with the same proof. Due to the negative result obtained in §8.2, we state it solely for almost everywhere and measure special hardness.

THEOREM 8.4. If $\mathscr{O}$ is almost everywhere special hard for $\Pi^0_n(\mathbf{2}^{<\omega})$ then it is also almost everywhere special hard for $\Sigma^0_{n+1}(\mathbf{2}^{<\omega})$. Idem with measure hardness.

PROBLEM 8.5. Do there exist almost everywhere or measure special hard families for Σ^0_n, $n \geq 3$?

§9. Theorems for n-randomness.

Using the ideas developed in [6], we extend the proof of the Pattern Theorem 6.1 of Part 1 [5] to outputs in $\mathfrak{P}(\mathbb{N})$ in three different contexts, correponding to the three types of computable maps introduced in §2.2. Thus, we present three variants of a basic randomness theorem. The three have a common definability hypothesis in a given set $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ (or $\mathfrak{P}(\mathbb{N}^d)^\ell$), and vary in the hardness hypothesis on $\mathscr{O}$: effective Wadge hardness, second order many one hardness, and special hardness. It turns out that in all three cases the weaker hypothesis of measure hardness suffices, so that we state these theorems with this hypothesis. To simplify notations, we state these theorems for $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$. Of course, they are still valid with $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N}^d)^\ell$.

The main randomness results of this paper are obtained as applications of the three variants of the basic theorem.

NOTATION 9.1. We assume Martin-Löf's definition of randomness (or its equivalent counterpart in terms of program-size complexity). For $n \geq 1$, n-randomness is randomness relative to oracle $\emptyset^{(n-1)}$, so that 1-randomness is usual Martin-Löf randomness. We also use Π^1_1 Martin-Löf randomness (which is simply obtained by replacing c.e. by Π^1_1 in the definition of Martin-Löf randomness) and some results obtained by Hjort & Nies in [11] about this notion.

9.1. Basic randomness theorem for $\Omega^\infty_U[\mathscr{O}]$.

THEOREM 9.2 (Basic randomness theorem: infinite inputs). Let $U : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ be a computable total map universal by adjunction and $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ and $n \geq 1$. Suppose that $\mathscr{O}$ is Scott Σ^0_n in $\mathfrak{P}(\mathbb{N})$ and measure Wadge hard for the class of open subsets of $\mathbf{2}^\omega$ of the form $X\mathbf{2}^\omega$ where X is $\Sigma^0_n(\mathbf{2}^{<\omega})$. Then the reals

$$\Omega^\infty_U[\mathscr{O}] = \mu(U^{-1}(\mathscr{O})), \qquad \Omega^\infty_U[\overline{\mathscr{O}}] = \mu(U^{-1}(\overline{\mathscr{O}}))$$

(the probabilities that an input in $\mathbf{2}^\omega$ is mapped by U into $\mathscr{O}$ and outside $\mathscr{O}$) are n-random respectively with Σ^0_n and Π^0_n left cuts (in the set of rational numbers).

Idem with Π^1_1 in place of Σ^0_n.

PROOF. First, Theorem 3.6 insures that $U^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^\omega)$, hence is Lebesgue measurable. Thus, we can consider its measure.

Consider an optimal (in the sense of the invariance theorem for Kolmogorov prefix complexity) prefix-free partial $\emptyset^{(n-1)}$-computable function $\varphi^{(n-1)} : \mathbf{2}^{<\omega} \to \mathbf{2}^{<\omega}$. The oracular version of Chaitin's celebrated theorem [9] insures that $\Omega_{\varphi^{(n-1)}} = \mu(dom(\varphi^{(n-1)})\mathbf{2}^\omega)$ is n-random with Σ^0_n left cut. Since the domain of $\varphi^{(n-1)}$ is $\Sigma^0_n(\mathbf{2}^{<\omega})$, the assumed measure Wadge hardness of $\mathscr{O}$ insures that there is some computable total map $F : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ and computable reals a, b such that

$$\mu(F^{-1}(\mathscr{O})) = a\mu(dom(\varphi^{(n-1)})\mathbf{2}^\omega) + b = a\Omega_{\varphi^{(n-1)}} + b. \tag{7}$$

Since U is universal by adjunction, there exists $i \in \mathbb{N}$ such that $F = U_i$. In particular, $F^{-1}(\mathscr{O}) = U_i^{-1}(\mathscr{O}) = \{\alpha \in \mathbf{2}^\omega \mid 0^i 1\alpha \in U^{-1}(\mathscr{O})\}$ so that

$$U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^\omega = 0^i 1\, F^{-1}(\mathscr{O}). \tag{8}$$

Using (7) and (8), we get $\mu(U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^\omega) = 2^{-i-1}(a\Omega_{\varphi^{(n-1)}} + b)$. Finally,

$$\begin{aligned} U^{-1}(\mathscr{O}) &= (U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^\omega) \,\cup\, (U^{-1}(\mathscr{O}) \setminus 0^i 1\mathbf{2}^\omega), \\ \mu(U^{-1}(\mathscr{O})) &= \mu(U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^\omega) + \mu(U^{-1}(\mathscr{O}) \setminus 0^i 1\mathbf{2}^\omega) \\ &= 2^{-i-1}(a\Omega_{\varphi^{(n-1)}} + b) + \mu(U^{-1}(\mathscr{O}) \setminus 0^i 1\mathbf{2}^\omega). \end{aligned}$$

Since $\Omega_{\varphi^{(n-1)}}$ is n-random with Σ^0_n left cut, so is $2^{-i-1}(a\Omega_{\varphi^{(n-1)}} + b)$. Also, since $U^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^\omega)$, so is $U^{-1}(\mathscr{O}) \setminus 0^i 1\mathbf{2}^\omega$ and its measure is a real with Σ^0_n left cut ([8], cf. also [5] Prop.3.2). It is known that the sum of two reals with Σ^0_n left cuts is n-random whenever one of them is n-random ([8], cf. also [5] Prop.3.6 or Downey & Hirschfeldt's book [10]); we conclude that $\mu(U^{-1}(\mathscr{O})) = \Omega^\infty_U[\mathscr{O}]$, is n-random with Σ^0_n left cut.

As for the complement $\overline{\mathscr{O}}$, observe that, since U is a total map, $\Omega^\infty_U[\overline{\mathscr{O}}] = 1 - \Omega^\infty_U[\mathscr{O}]$ hence is n-random with Π^0_n left cut.

In case $\mathscr{O}$ is Scott Π^1_1 in $\mathfrak{P}(\mathbb{N})$ and Π^1_1 measure Wadge hard, the above argument goes through using

- the Π^1_1 version $\varphi^{(\Pi^1_1)}$ of a universal map $\varphi^{(n-1)}$ for self-delimited partial $\emptyset^{(n-1)}$-computable maps $\mathbf{2}^{<\omega} \to \mathbf{2}^{<\omega}$ and the associated Π^1_1 version of Chaitin's $\Omega_{\varphi^{(n-1)}}$ (cf. Hjort & Nies, [11]) which is a Martin-Löf Π^1_1 random real with Π^1_1 left cut,
- the extension to Π^1_1 of the fact that the sum of two n-random reals with Σ^0_n left cut is n-random. ⊣

REMARK 9.3. In the above theorem, and also in Theorem 9.5 below, the assumption $\mathscr{O}$ is Scott Σ^0_n in $\mathfrak{P}(\mathbb{N})$ can be weakened to $U^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^\omega)$.

9.2. Applications to n-randomness with $\Omega^\infty_U[\mathscr{O}]$.

THEOREM 9.4 (n-randomness with $\Omega^\infty_U[\mathscr{O}]$). Let $U : \mathbf{2}^\omega \to \mathfrak{P}(\mathbb{N})$ be a computable total map universal by adjunction. Then, for each line of the $\Omega^\infty_U[\mathscr{O}]$

column of Table 1, the real $\Omega_U^{\infty}[\mathscr{O}]$ is n-random (or Π_1^1-random) with Σ_n^0 or Π_n^0 (or Π_1^1) left cut as stated.

Proof. Observe that the conditions of application of Theorem 9.2 are fulfilled for every line stating some randomness result for $\Omega_U^{\infty}[\mathscr{O}]$. In fact, the Scott definability of $\mathscr{O}$ is given by Propositions 4.1 and 4.7. The Wadge or almost everywhere Wadge hardness of $\mathscr{O}$ is given by Propositions 7.2 and 7.4, Theorems 7.7 and 7.18 and Proposition 7.21. ⊣

9.3. Basic randomness theorem for $\Omega_U^{\infty}[k,\mathscr{O}]$.

Theorem 9.5 (Basic theorem: large enough finite inputs). Let $U : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ be a computable total map universal by adjunction and $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$ and $n \geq 1$. Suppose that $\mathscr{O}$ is Scott Σ_n^0 in $\mathfrak{P}(\mathbb{N})$ and measure second order many-one hard for the class of $\Sigma_n^0(\mathbf{2}^{<\omega})$ sets of words. Then, for k large enough, the reals

$$\Omega_U^{\infty}[k,\mathscr{O}] = \mu((U^{-1}(\mathscr{O}) \cap 2^{\geq k})\mathbf{2}^{\omega}), \qquad \Omega_U^{\infty}[k,\overline{\mathscr{O}}] = \mu((U^{-1}(\overline{\mathscr{O}}) \cap 2^{\geq k})\mathbf{2}^{\omega})$$

(the probabilities that an element in $\mathbf{2}^{\omega}$ has at least one length $\geq k$ prefix mapped by U into $\mathscr{O}$ and outside $\mathscr{O}$) are respectively n-random and $(n+1)$-random with Σ_n^0 and Σ_{n+1}^0 left cuts.

If $\mathscr{O}$ is Scott Π_1^1 in $\mathfrak{P}(\mathbb{N})$ and measure second order many-one hard for the class of $\Pi_1^1(\mathbf{2}^{<\omega})$, the real $\Omega_U^{\infty}[k,\mathscr{O}]$ is Martin-Löf Π_1^1 random with Π_1^1 left cut.

Problem 9.6. In the Π_1^1 case, we do not know whether and how much $\Omega_U^{\infty}[k,\overline{\mathscr{O}}]$ is random.

Proof. We argue as in the proof of Theorem 9.2 with a complementary argument in the vein of that developed in our paper [6]. Using the assumed measure second order many-one hardness of $\mathscr{O}$, let $F : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ be a computable total map such that $\mu(F^{-1}(\mathscr{O})\mathbf{2}^{\omega}) = a\ \mu(dom(\varphi^{(n-1)})\mathbf{2}^{\omega}) + b$ where a, b are computable reals, $a \neq 0$. Using universality by adjunction of U, let $i \in \mathbb{N}$ be such that $F = U_i$. Then $F^{-1}(\mathscr{O}) = U_i^{-1}(\mathscr{O}) = \{u \in \mathbf{2}^{<\omega} \mid 0^i 1u \in U^{-1}(\mathscr{O})\}$, so that $U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^{<\omega} = 0^i 1\, F^{-1}(\mathscr{O})$ and

$$\mu((U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^{<\omega})\mathbf{2}^{\omega}) = \mu(0^i 1\, F^{-1}(\mathscr{O})\mathbf{2}^{\omega}) = 2^{-i-1}(a\Omega_{\varphi^{(n-1)}} + b).$$

Thus, $\mu((U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^{<\omega})\mathbf{2}^{\omega})$ is n-random. Since $dom(\varphi^{(n-1)})$ is prefix-free so are $F^{-1}(\mathscr{O})$ and $U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^{<\omega}$. For $k \geq i$, we have

$$U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^{<\omega} = U^{-1}(\mathscr{O}) \cap 0^i 12^{\geq k-i-1}\ \cup\ U^{-1}(\mathscr{O}) \cap 0^i 12^{<k-i-1}.$$

The two sets on the right are disjoint and their union is prefix-free, therefore $(U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^{<\omega})\mathbf{2}^{\omega}$ is the disjoint union of $(U^{-1}(\mathscr{O}) \cap 0^i 12^{\geq k-i-1})\mathbf{2}^{\omega}$ and $(U^{-1}(\mathscr{O}) \cap 0^i 12^{<k-i-1})\mathbf{2}^{\omega}$. So that

$$\begin{aligned}\mu((U^{-1}(\mathscr{O}) \cap 0^i 1\mathbf{2}^{<\omega})\mathbf{2}^{\omega}) &= \mu((U^{-1}(\mathscr{O}) \cap 0^i 12^{\geq k-i-1})\mathbf{2}^{\omega}) \\ &\quad + \mu((U^{-1}(\mathscr{O}) \cap 0^i 12^{<k-i-1})\mathbf{2}^{\omega}).\end{aligned}$$

Now, $U^{-1}(\mathscr{O}) \cap 0^i 12^{<k-i-1}$ is finite, so that the second term on the right is dyadic rational. As a consequence, the real $\mu((U^{-1}(\mathscr{O}) \cap 0^i 12^{\geq k-i-1})\mathbf{2}^{\omega})$ is n-random.

Finally, $(U^{-1}(\mathscr{O}) \cap 2^{\geq k})\mathbf{2}^{\omega}$ is the disjoint union of $(U^{-1}(\mathscr{O}) \cap 0^i 1 2^{\geq k-i-1})\mathbf{2}^{\omega}$ and $(U^{-1}(\mathscr{O}) \cap \{v \in 2^{\geq k} \mid 0^i 1 \not\leq_{\mathrm{pref}} v\})\mathbf{2}^{\omega}$. Thus,

$$\mu((U^{-1}(\mathscr{O}) \cap 2^{\geq k})\mathbf{2}^{\omega}) = \mu((U^{-1}(\mathscr{O}) \cap 0^i 1 2^{\geq k-i-1})\mathbf{2}^{\omega}) + \mu((U^{-1}(\mathscr{O}) \cap \{v \in 2^{\geq k} \mid 0^i 1 \not\leq_{\mathrm{pref}} v\})\mathbf{2}^{\omega}).$$

The two reals on the right side have Σ^0_n left cuts. Since the first one is n-random, so is their sum. Thus, $\mu((U^{-1}(\mathscr{O}) \cap 2^{\geq k})\mathbf{2}^{\omega}) = \Omega^{\propto}_U[k, \mathscr{O}]$ is n-random with Σ^0_n left cut for any $k \geq i$.

As for the complement $\overline{\mathscr{O}}$, apply Proposition 5.8 and Theorem 5.11.

As for Π^1_1, argue as in the proof of Theorem 9.2. ⊣

9.4. Application to n-randomness with $\Omega^{\propto}_U[k, \mathscr{O}]$.

THEOREM 9.7 (n-randomness with $\Omega^{\propto}_U[k, \mathscr{O}]$). Let $U : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ be a computable total map universal by adjunction. Then, for each line of the $\Omega^{\propto}_U[k, \mathscr{O}]$ column of Table 1, the real $\Omega^{\propto}_U[k, \mathscr{O}]$ is n-random (or Π^1_1-random), for k large enough, with Σ^0_n or Π^0_n (or Π^1_1) left cut as stated.

PROOF. Observe that the conditions of application of Theorem 9.5 are fulfilled for every line stating some randomness result for $\Omega^{\propto}_U[k, \mathscr{O}]$. In fact, the Scott definability of $\mathscr{O}$ is given by Propositions 4.1 and 4.7. The second order many-one hardness of $\mathscr{O}$ is given by Propositions 6.2, 6.6 and 6.10. ⊣

9.5. Basic randomness theorem with $\Omega^{\bowtie}_U[\mathscr{O}]$.

THEOREM 9.8 (Basic theorem: self-delimited finite inputs). Let $U : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ be a self-delimited partial computable map universal by adjunction and $\mathscr{O} \subseteq \mathfrak{P}(\mathbb{N})$. Suppose $n \geq 2$ and $\mathscr{O}$ is Scott Σ^0_n in $\mathfrak{P}(\mathbb{N})$ and measure special hard for the class of $\Sigma^0_n(\mathbf{2}^{<\omega})$ sets of words. Then the real

$$\Omega^{\bowtie}_U[\mathscr{O}] = \mu(U^{-1}(\mathscr{O})\mathbf{2}^{\omega})$$

(the probability that an element in $\mathbf{2}^{\omega}$ has some prefix in the domain of U which is mapped by U into $\mathscr{O}$) is n-random with Σ^0_n left cut.

Idem with Π^1_1 in place of Σ^0_n.

In case $n \geq 3$, the real $\Omega^{\bowtie}_U[\overline{\mathscr{O}}] = \mu((U^{-1}(\overline{\mathscr{O}})\mathbf{2}^{\omega})$ (the probability that an element in $\mathbf{2}^{\omega}$ has some prefix in the domain of U which is mapped by U outside $\mathscr{O}$) is n-random with Π^0_n left cut.

PROOF. Similar to the proof of Theorem 9.2, with $F^{-1}(\mathscr{O})\mathbf{2}^{\omega}$ and $U^{-1}(\mathscr{O})\mathbf{2}^{\omega}$ in place of $F^{-1}(\mathscr{O})$ and $U^{-1}(\mathscr{O})$. Prefix-freeness of the domain of F (hence of U_i) allows all arguments to go through. In order to see that $U^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^{<\omega})$ and $U^{-1}(\overline{\mathscr{O}})$ is $\Pi^0_n(\mathbf{2}^{<\omega})$, the application of Theorem 3.6 requires respectively conditions $n \geq 2$ and $n \geq 3$.

As for the complement $\overline{\mathscr{O}}$, observe that $\Omega^{\bowtie}_U[\overline{\mathscr{O}}] = \Omega^{\bowtie}_U[\mathfrak{P}(\mathbb{N})] - \Omega^{\bowtie}_U[\mathscr{O}]$. Applying the theorem with $\mathscr{O} = \mathfrak{P}(\mathbb{N})$, which is Scott Σ^0_1 hence also Scott Σ^0_2 and open special hard for $\Sigma^0_2(\mathbf{2}^{<\omega})$ (Proposition 8.1), we see that $\Omega^{\bowtie}_U[\mathfrak{P}(\mathbb{N})]$ is 2-random with Σ^0_2 left cut. If $n \geq 3$, the above equation insures that $\Omega^{\bowtie}_U[\overline{\mathscr{O}}\mathbf{2}^{\omega}]$ is n-random with Σ^0_n left cut.

As for Π^1_1, argue as in the proof of Theorem 9.2. ⊣

REMARK 9.9. In the above theorem, the assumption $\mathscr{O}$ is Scott Σ^0_n in $\mathfrak{P}(\mathbb{N})$ can be weakened to $U^{-1}(\mathscr{O})$ is $\Sigma^0_n(\mathbf{2}^{<\omega})$. In this case, conditions $n \geq 2$ or $n \geq 3$ can be removed.

REMARK 9.10. In case $\mathscr{O}$ is Σ^0_2 and contains some c.e. set, then $\Omega^{\bowtie}_U[\overline{\mathscr{O}}] = \Omega^{\bowtie}_U[\mathfrak{P}(\mathbb{N})] - \Omega^{\bowtie}_U[\mathscr{O}]$ is the difference of two reals with Σ^0_2 left cuts. Rettinger's Theorem (cf. Downey & Hirschfeldt's book, §8.5.2) insures that such a real cannot be 2-random except if its left cut is Σ^0_2 or Π^0_2.

Although the 2-randomness result with $\Omega^{\bowtie}_U[\mathscr{O}]$ given by Theorem 9.11 is far more powerful than those for 2-randomness with $\Omega^{\infty}_U[k,\mathscr{O}]$ (§9.4) and with $\Omega^{\infty}_U[\mathscr{O}]$ (§9.2), the methods developed in this paper do not lead to any n-randomness result for $n \geq 3$ with $\Omega^{\bowtie}_U[\mathscr{O}]$. The reason is Proposition 8.3 and Problem 8.5.

9.6. Application to 2-randomness with $\Omega^{\bowtie}_U[\mathscr{O}]$. The following theorem with $\Omega^{\bowtie}_U[\mathscr{O}]$ generalizes Theorem 1.10 of [5] to the $\mathfrak{P}(\mathbb{N})$ context.

THEOREM 9.11 (2-randomness with $\Omega^{\bowtie}_U[\mathscr{O}]$). Let $U : \mathbf{2}^{<\omega} \to \mathfrak{P}(\mathbb{N})$ be a self-delimited partial computable map universal by adjunction.

1. If $\mathscr{O}$ is Scott Σ^0_1 in $\mathfrak{P}(\mathbb{N})$ and contains some computably enumerable set of integers then the reals $\Omega^{\bowtie}_U[\mathscr{O}]$ and $\Omega^{\bowtie}_U[\overline{\mathscr{O}}]$ are both 2-random with Σ^0_2 left cuts.
2. If $\mathscr{O}$ is Scott Σ^0_2 in $\mathfrak{P}(\mathbb{N})$ and contains some computably enumerable set of integers then the real $\Omega^{\bowtie}_U[\mathscr{O}]$ is 2-random with Σ^0_2 left cut.

This justifies all randomness results mentioned in the $\Omega^{\bowtie}_U[\mathscr{O}]$ column of Table 1.

PROOF. Observe that the conditions of application of Theorem 9.8 are fulfilled for every line stating some randomness result for $\Omega^{\bowtie}_U[\mathscr{O}]$. In fact, the Scott definability of $\mathscr{O}$ is given by Propositions 4.1 and 4.7. The second order open special hardness of $\mathscr{O}$ for Σ^0_2 is given by Proposition 8.1. which applies in case $\mathscr{O}$ or is Scott Σ^0_1, Π^0_1, $\Sigma^0_1 \wedge \Pi^0_1$ or Σ^0_2,

As for lines 9a, 9b, 9c, observe that $\Omega^{\bowtie}_U[\mathscr{O}] = \Omega^{\bowtie}_U[\widetilde{\mathscr{O}}]$ whenever $\mathscr{O}$ and $\widetilde{\mathscr{O}}$ have the same c.e. sets. Letting $\widetilde{\mathscr{O}} = \mathfrak{P}(\mathbb{N})$, the $\Omega^{\bowtie}_U[\mathscr{O}]$ randomness results for lines 9a, 9b, 9c reduces to that of line 0. ⊣

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