

Randomness and uniform distribution modulo one

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Abstract

We elaborate the notions of Martin-Löf and Schnorr randomness for real numbers in terms of uniform distribution of sequences. We give a necessary condition for a real number to be Schnorr random expressed in terms of classical uniform distribution of sequences. This extends the result proved by Avigad for sequences of linear functions with integer coefficients to the wider classical class of Koksma sequences of functions. And, by requiring equidistribution with respect to every computably enumerable open set (respectively, computably enumerable open set with computable measure) in the unit interval, we give a sufficient condition for Martin-Löf (respectively Schnorr) randomness.

Keywords: uniform distribution modulo 1, Martin-Löf randomness, random real number, Koksma General Metric Theorem.

1 From randomness to uniform distribution

How is the notion of randomness from algorithmic information theory related to the notion of uniform distribution from number theory? In this paper we elaborate the definitions of Martin-Löf randomness and Schnorr randomness for real numbers in terms of uniform distribution of sequences.

1.1 Randomness

Intuitively, given a probability measure such as Lebesgue measure, a real number is random if it has the properties of almost all real numbers; that is, if it belongs to no set of measure zero. The intuition can not be taken literally because every real number belongs to the singleton set containing it, which has measure zero. To prevent the property being trivial one can restrict to computably defined sets, as done by Martin-Löf in [21] and Schnorr in [23, 12].

A Martin-Löf test for randomness is a sequence $(U_n)_{n \in \mathbb{N}}$ of decreasing open sets $U_n = \bigcup_{k \in \mathbb{N}} I_{n,k}$ where $(I_{n,k})_{n,k \in \mathbb{N}}$ is a computable sequence of open intervals with rational endpoints and such that the sequence of Lebesgue measures $(\lambda(U_n))_{n \in \mathbb{N}}$ is upper-bounded by a computable function which converges to zero. A real number passes this test if it is not in all U_n 's. A real number is Martin-Löf random if it passes every such test, i.e. if it is outside every one of these particular measure zero sets $\bigcap_{n \in \mathbb{N}} U_n$. To get Schnorr randomness, consider those Martin-Löf tests such that the sequence of reals $(\lambda(U_n))_{n \in \mathbb{N}}$ is computable. Martin-Löf random reals constitute a proper subset of Schnorr random reals. Since there are only countably many of these tests, there are also only countably many of these measure zero sets and this implies that almost all real numbers, with respect to Lebesgue measure, are Martin-Löf random and, a fortiori, Schnorr random.

The definition entails that the fractional expansions of Martin-Löf random (or Schnorr random) real numbers obey all the usual probability laws. And it follows that all the computable real numbers, such as the irrational algebraic numbers and the usual mathematical constants including π and e , are not Martin-Löf random nor Schnorr random. Chaitin [11] showed that the halting probability of any universal Turing machine with prefix-free domain is Martin-Löf random. Many machine behaviours have Martin-Löf random probabilities, for instance see [4, 6, 7, 5, 3, 2]. For a presentation of the theory of randomness see [13, 22].

1.2 Uniform distribution modulo 1

An infinite sequence $(x_n)_{n \geq 1}$ of real numbers is uniformly distributed modulo 1, abbreviated u.d. mod 1, if the sequence formed by the fractional parts of its terms is equidistributed in the unit interval. This means that for each subinterval of the unit interval, asymptotically, the proportion of terms falling within that subinterval is equal to its length. For a real number x , we write $[x]$ to denote its integer part and $\{x\} = x - [x]$ for its fractional expansion.

Thus, a sequence $(x_n)_{n \geq 1}$ of real numbers is u.d. mod 1 if for every half open interval $[a, b)$ in $[0, 1]$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n : 1 \leq n \leq N \text{ and } \{x_n\} \in [a, b)\} = b - a. \quad (1)$$

A presentation of the theory of uniform distribution can be read from [20, 14, 10].

In the years 1909 and 1910, Bohl, Sierpiński and Weyl independently established that a number x is irrational exactly when the sequence $(nx)_{n \geq 1}$ is u.d. mod 1 (cf. [20, p.8 Example 2.1 and p.21 Notes]). The subset of the irrational numbers that Borel called absolutely normal¹ [9] also has a characterization in terms of uniform distribution: a real number is absolutely normal exactly when, for every integer t greater than 1, the sequence $(t^n x)_{n \geq 1}$ is u.d. mod 1, see [20][p.70 Theorem 8.1 and p.74 Notes] and [10].

The Martin-Löf random and Schnorr random real numbers are a proper subset of the irrational numbers and also of the absolutely normal numbers, we aim to find a class of sequences of functions $(u_n : [0, 1] \rightarrow \mathbb{R})_{n \geq 1}$ such that x is Martin-Löf random exactly when $(u_n(x))_{n \geq 1}$ is u.d. mod 1.

The following observation gives a first delimitation of what is possible in terms of linear functions.

Proposition 1. *For every real x there exists a strictly monotone increasing sequence $(a_n)_{n \geq 1}$ of positive integers such that the sequence $(2^{a_n} x)_{n \geq 1}$ is not u.d. Moreover, if x is computable then so is the sequence $(a_n)_{n \geq 1}$.*

Proof. If x is dyadic, says $x = p/2^k$ with $p \in \mathbb{N}$ then the fractional part of $2^\ell x$ is 0 for every $\ell \geq k$, hence the left-hand side of equality (1) is at most k/N if $0 < a < b$ and a fortiori smaller than $b - a$ for $N > k/(b - a)$. In particular, the sequence $(2^\ell x)_{\ell \geq k}$ is not u.d.

If x is not dyadic, let the fractional part of x be $\sum_{n \geq 1} \delta_n / 2^n$ where the δ_n 's are the binary digits of x . Infinitely many of these digits are 0 and infinitely many of them are 1. Let $(a_k)_{k \geq 1}$ be the sequence of ranks (in $\mathbb{N} \setminus \{0\}$) of those which are 0, i.e. $\delta_{a_k} = 0$ for all $k \geq 1$ and $\delta_n = 1$ for all n different from all the a_k 's. Observe that the first digit of $2^{n-1}x$ is δ_n . Thus, the fractional part of $2^{a_k-1}x$ is in $[0, 1/2)$ for all $k \geq 1$ hence the left-hand side of equality (1) is 1 for all N and does not tend to $1/2$. This shows that the sequence $(2^{a_k-1}x)_{k \geq 1}$ is not u.d. Of course, if x is computable so is the sequence $(a_k)_{k \geq 1}$. \square

1.3 Koksma's General Metric theorem

We consider the class of sequences in Koksma's General Metric theorem [19] but restricted to a suitable computability condition. Koksma's General Metric Theorem, which can also be read from [20, Theorem 4.3 Chapter 1, p.34], introduces the class of so-called Koksma sequences of functions and establishes that if $(u_n)_{n \geq 1}$ is Koksma then, for almost all real numbers $x \in [0, 1]$, the sequence $(u_n(x))_{n \geq 1}$ is u.d. mod 1.

¹A real number is simply normal to an integer base b greater than or equal to 2 if in the fractional expansion of x in base b every digit in $\{0, 1, \dots, b-1\}$ occurs with the same asymptotic frequency $1/b$. A number is normal to base b if it is simply normal to the bases b, b^2, b^3, \dots . A number is absolutely normal if it is normal to all integer bases greater than or equal to 2.

Definition 2 (Koksma sequences of functions). Let $K > 0$. A sequence $(u_n)_{n \geq 1}$ of functions $u_n : [0, 1] \rightarrow \mathbb{R}$ is K -Koksma if

- the function u_n is continuously differentiable for every n ,
- the difference $u'_m(x) - u'_n(x)$ is a monotone function on $[0, 1]$ for all m, n ,
- $|u'_m(x) - u'_n(x)| \geq K$ for all $m \neq n$ and all $x \in [0, 1]$.

A sequence $(u_n)_{n \geq 1}$ is Koksma if it is K -Koksma for some K .

For example, if $t > 1$ and $u_n(x) = t^n x$ then $(u_n)_{n \geq 0}$ is $(t - 1)$ -Koksma. Similarly if $(a_n)_{n \geq 1}$ is a sequence of pairwise distinct integers and $u_n(x) = a_n x$ then $(u_n)_{n \geq 0}$ is 1-Koksma. In contrast, for $u_n(x) = x^n$, $(u_n)_{n \geq 0}$ is not Koksma and $u_n(x) = x + a$, for some a is not Koksma either. The importance of the Koksma class is stressed by the following classical result.

Proposition 3 (Koksma's General Metric theorem, 1935). *If the sequence of functions $(u_n)_{n \geq 1}$ is Koksma then the set of reals $x \in [0, 1]$ such that the sequence of reals $(u_n(x))_{n \geq 1}$ is u.d. mod 1 has Lebesgue measure 1.*

1.4 Effective Koksma uniformly distributed reals

The theory of computability formalizes the notion of function, or sequence of functions $\mathbb{N} \rightarrow \mathbb{N}$ or $[0, 1] \rightarrow \mathbb{R}$ which can be effectively computed by some algorithm. Using it, we can define *effective Koksma sequences* and *effective Koksma uniformly distributed reals*.

Definition 4 (Effective Koksma functions). A sequence $(u_n)_{n \geq 1}$ of functions is effective Koksma if it is Koksma and computable and if $(u'_n)_{n \geq 1}$ is also computable.

Definition 5 (Effective Koksma uniformly distributed reals). A real $x \in [0, 1]$ is effective Koksma uniformly distributed (in short effective Koksma u.d.), if, for every effective Koksma sequence $(u_n)_{n \geq 1}$, the sequence of reals $(u_n(x))_{n \geq 1}$ is u.d. mod 1.

Proposition 3 immediately implies the following result.

Proposition 6. *The set of effective Koksma u.d. reals $x \in [0, 1]$ has Lebesgue measure 1.*

1.5 Theorem 1: Randomness implies effective Koksma u.d.

Proposition 6 ensures that almost every Schnorr random real is effective Koksma u.d. In fact, almost everywhere can be replaced by everywhere.

Theorem 1. *Every Schnorr random real (a fortiori every Martin-Löf random real) is effective Koksma u.d.*

This theorem extends Avigad’s [1, Theorem 2.1], see also [17] and the references there. Indeed, Avigad considers “linear” Koksma sequences of functions, namely sequences of linear functions $(x \mapsto a_n x)_{n \geq 1}$ where $(a_n)_{n \geq 1}$ is any computable sequence of pairwise distinct positive integers (an assumption which trivially insures the effective Koksma condition). He proves that, for every Schnorr random real x , the sequence of reals $(a_n x)_{n \geq 1}$ is u.d. mod 1.

The proof of Theorem 1 is done in Section 4. It uses Koksma’s General Metric Theorem.

Remark 7. By Proposition 1 no computable real can be effective Koksma u.d. Since there are Martin-Löf random reals which are computable in \emptyset' , Theorem 1 ensures that there exist effective Koksma u.d. reals which are computable in \emptyset' .

1.6 Around Theorem 1

We do not know whether the converse fails or not, i.e. whether there exists an effective Koksma u.d. real which is not Schnorr random. For linear functions with coefficients in \mathbb{N} , Avigad [1] proves the following result.

First, consider the classical notion of Σ_1^0 sets.

Definition 8. A subset S of $[0, 1]$ is effectively open, or Σ_1^0 , if it is of the form $S = \bigcup_{k \geq 1} I_k$ for some computable sequence $(I_k)_{k \geq 1}$ of (possibly empty) open intervals I_k with rational endpoints. Complements of such sets are called effectively closed, or Π_1^0 . Schnorr Σ_1^0 sets are those Σ_1^0 sets with computable Lebesgue measure.

Proposition 9 (Avigad [1]). *There is an uncountable perfect Π_1^0 set $C \subset [0, 1]$ of Lebesgue measure 0 – hence disjoint from the set of Schnorr random or simply Kurtz random reals – and an atomless probability measure ν on C such that, for every sequence $(a_n)_{n \geq 1}$ of pairwise distinct positive integers, the set $\{x \in C : (a_n x)_{n \geq 1} \text{ is u.d.}\}$ has ν measure 1. In particular, there are reals which are not Schnorr random nor Kurtz random and such that $(a_n x)_{n \geq 1}$ is u.d. for every computable sequence $(a_n)_{n \geq 1}$ of pairwise distinct positive integers.*

Independently of the effective character of this result, it is related to the existence of a measure with positive Fourier dimension, cf. [15][Corollary 4, p.73]. We do not know whether the techniques around Koksma’s General Metric Theorem and the Fourier dimension theory can be pushed to extend Proposition 9 by replacing sequences of linear functions $(x \mapsto a_n x)_{n \geq 1}$ by general Koksma sequences of functions, which would ensure that the converse of Theorem 1 fails.

2 From Σ_1^0 -uniform distribution to randomness

To capture Martin-Löf and Schnorr randomness we strengthen the classical definition of uniform distribution. With this strengthening, in Theorem 2 we characterize Martin-Löf and Schnorr randomness in terms of uniform distribution modulo 1.

2.1 \mathcal{C} -uniform distribution

A natural extension of uniform distribution is to replace intervals by a larger family of subsets of $[0, 1]$. Recall that λ denotes Lebesgue measure.

Definition 10 (\mathcal{C} -u.d. mod 1). Let \mathcal{C} be a class of measurable subsets of $[0, 1]$. A sequence $(x_n)_{n \geq 1}$ of real numbers is \mathcal{C} -uniformly distributed modulo 1, abbreviated \mathcal{C} -u.d., if for every $A \in \mathcal{C}$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n : 1 \leq n \leq N, \{x_n\} \in A\} = \lambda(A). \quad (2)$$

The following two remarks delimitate some boundaries of the notion.

Remark 11 (When \mathcal{C} is too small the notion adds nothing new). In case \mathcal{C} is the family of the rational intervals, or it is the family of all Jordan-measurable subsets, or it is the family of λ -continuous sets (which are measurable sets whose boundary has measure 0, see [20, Chapter 1, p.5-6]), then \mathcal{C} -uniform distribution is exactly uniform distribution.

Remark 12 (When \mathcal{C} is too large the notion is vacuous). If the class \mathcal{C} contains all open subsets of $[0, 1]$ then there is no sequence $(x_n)_{n \geq 1}$ that is \mathcal{C} -u.d. Indeed, letting

$$A = [0, 1) \cap \bigcup_{n \geq 1} (\{x_n\} - 2^{-n-2}, \{x_n\} + 2^{-n-2})$$

we have $\lambda(A) \leq 1/2$ whereas the left-hand side of equality (2) is 1.

On the other hand, the next proposition shows that the notion is not trivial if \mathcal{C} is any countable family. A sequence $(x_n)_{n \geq 1}$ of real numbers in the unit interval can be viewed as a point in the Cartesian product $[0, 1]^{\mathbb{N}}$. The Lebesgue measure λ on $[0, 1]$ induces the product measure λ_{∞} on $[0, 1]^{\mathbb{N}}$.

Proposition 13. *Let \mathcal{C} be a countable class of measurable subsets of $[0, 1]$. Then λ_{∞} -almost all elements in $[0, 1]^{\mathbb{N}}$ are \mathcal{C} -u.d.*

Remark 14. Proposition 13 is an easy variant of a classical result due to Hlawka, 1956 (see [20, Theorem 2.2 Chapter 3 p.183]). Let μ be a probability measure on the Borel subsets of a compact space X having a countable basis. Then $\mu_{\infty}(S) = 1$ where S is the set of sequences $(\{x_n\})_{n \in \mathbb{N}}$ of elements of X such that, for every real valued continuous function f defined on X , we have $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{x_n\}) = \int_X f$.

Proof of Proposition 13. By Weyl's Criterion (see [10, Theorem 1.2] or [20, Theorem 1.1 Chapter 1, p.2]), a sequence $(x_n)_{n \geq 1}$ of real numbers is u.d. if, and only if, for every real valued continuous function f defined on $[0, 1]$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\{x_n\}) = \int_0^1 f(x) dx. \quad (3)$$

We use a result which follows from [20, Lemma 2.1 Chapter 3 p.182], where it is stated for bounded Borel functions but its proof applies as well to bounded

measurable functions $[0, 1] \rightarrow \mathbb{R}$, hence it applies to our case: if $f : [0, 1] \rightarrow \mathbb{R}$ is a bounded measurable function then, for λ_∞ -almost every sequence $(x_n)_{n \geq 1}$, equality (3) holds (i.e. the mean value of f on the x_n 's exists and is equal to the integral of f).

The case where \mathcal{C} consists of just one measurable set A is solved by the result above, letting f be the characteristic function of A . By countable additivity of λ_∞ , we get the case where \mathcal{C} is countable. \square

2.2 Σ_1^0 -uniform distribution

Introducing a computability constraint, the following definition avoids the pitfall of Remark 12 and benefits from Proposition 13. Since the class Σ_1^0 (see Definition 8) is a countable family of subsets in $[0, 1]$, Proposition 13 ensures that almost all sequences $(x_n)_{n \geq 1}$ of real numbers are Σ_1^0 -u.d.

Definition 15. Let Schnorr Σ_1^0 -u.d. mean \mathcal{C} -u.d. where \mathcal{C} is the family of Σ_1^0 subsets of $[0, 1]$ having computable measure.

Clearly, Σ_1^0 -u.d. — hence also Schnorr Σ_1^0 -u.d. — imply plain u.d. Let's see that the notions are different.

Proposition 16. *There is a sequence $(x_n)_{n \geq 1}$ that is u.d. but not Schnorr Σ_1^0 -u.d.*

Proof. Let x be a computable and irrational real number. As mentioned in §1.2, since x is irrational the sequence $(nx)_{n \geq 1}$ is u.d.; we show that it is not Schnorr Σ_1^0 -u.d. Let $(b_{n,j})_{j \geq 1}$ be the expansion of nx in base 2, for $n = 1, 2, \dots$. Let S be the following subset of $[0, 1]$,

$$S = \bigcup_{n \geq 1} S_n \quad \text{where } S_n = (0.b_{n,1}b_{n,2} \dots b_{n,2n}0, 0.b_{n,1}b_{n,2} \dots b_{n,2n}1)$$

Since x is computable, S is an open computably enumerable subset of $[0, 1]$. Also, it has computable measure: the dyadic rational $\lambda(\bigcup_{n < p} S_n)$ approximating $\lambda(S)$ up to $2^{-2p+2}/3$ since

$$\lambda\left(\bigcup_{n \geq p} S_n\right) \leq \sum_{n \geq p} 2^{-2n} = 2^{-2p+2}/3.$$

By construction, for each $n \geq 1$ the real nx is in the n -th interval defining S , so that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n : 1 \leq n \leq N, \{nx\} \in S\} = 1.$$

Since

$$\lambda(S) \leq \sum_{n \geq 1} 2^{-2n} = 1/3 < 1$$

Schnorr Σ_1^0 -u.d. fails. \square

2.3 Theorem 2: From Σ_1^0 -u.d. to randomness

We consider the following classical notion.

Definition 17 (Lipschitz function). For $\ell > 0$, a function $f : [0, 1] \rightarrow \mathbb{R}$ is ℓ -Lipschitz if $|f(x) - f(y)| \leq \ell|x - y|$ for every $x, y \in [0, 1]$.

Theorem 2 (From Σ_1^0 -u.d. to randomness). *Let $(u_n)_{n \geq 1}$ be a computable sequence of functions $[0, 1] \rightarrow \mathbb{R}$ which is computably Lipschitz, i.e. $(\ell_n)_{n \geq 1}$ -Lipschitz for some computable sequence $(\ell_n)_{n \geq 1}$ of rationals. If the real $x \in [0, 1]$ is such that $(u_n(x))_{n \geq 1}$ is Σ_1^0 -u.d. (respectively Schnorr Σ_1^0 -u.d.) then x is Martin-Löf (respectively Schnorr) random.*

Remark 18. By the mean value theorem, every function $u : [0, 1] \rightarrow \mathbb{R}$ continuously differentiable is ℓ -Lipschitz for any $\ell \geq \max_{t \in [0, 1]} |u'(t)|$. In particular, the class of computable sequences of computably Lipschitz functions $[0, 1] \rightarrow \mathbb{R}$ is far wider than the Koksma class.

The proof of Theorem 2 is done in Section 5.

3 Characterization of randomness with Σ_1^0 -u.d.

3.1 From randomness to Σ_1^0 -u.d.

Let's consider these classical notions and results on measure-preserving ergodic operators.

Definition 19. 1. Let $T : [0, 1] \rightarrow [0, 1]$. A set $A \subseteq [0, 1]$ is almost invariant if A and $T^{-1}(A)$ coincide up to a Lebesgue measure 0 set.

2. A measure-preserving operator $T : [0, 1] \rightarrow [0, 1]$ is ergodic if every almost invariant set has Lebesgue measure 0 or 1.

Remark 20. Two simple examples of ergodic maps on $[0, 1]$ are the shift $x \mapsto 2x \bmod 1$ (corresponding to the shift on the Cantor space $2^{\mathbb{N}}$), and the translation $x \mapsto x + a \bmod 1$ when a is irrational (cf. [24], Chap. 5, Example p.408). Observe that, in this case, the continuity implied by computability is relative to the topology of the disk: a typical neighborhood of 0 is a set $[0, \theta) \cup (1 - \theta, 1)$.

Proposition 21 (Birkhoff and Khinchin Theorem, 1931). *If $T : [0, 1] \rightarrow [0, 1]$ is measure-preserving and ergodic and $A \subseteq [0, 1]$ is Lebesgue measurable then for almost all $x \in [0, 1]$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \left\{ n : 1 \leq n \leq N, T^n(x) \in A \right\} = \lambda(A).$$

A decade ago this theorem has been effectivized, first by P. Gács et al. [18], 2011 for Schnorr randomness, and then by L. Bienvenu et al. [8, Theorem 8] and by J. Franklin et al. [16, Theorem 6], 2012 for Martin-Löf randomness.

Proposition 22 (Effective Birkhoff Ergodic Theorem[18, 8, 16]). *Let T be a computable ergodic operator $[0, 1) \rightarrow [0, 1)$ and let $U \subseteq [0, 1)$ be a Σ_1^0 (respectively Σ_1^0 with computable measure) set. For every Martin-Löf (respectively Schnorr) random real $x \in [0, 1)$ we have*

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \#\{n: 1 \leq n \leq N, T^n(x) \in U\} = \lambda(U).$$

In other words, if x is Martin-Löf (respectively Schnorr) random then the sequence $(T^n(x))_{n \geq 1}$ is Σ_1^0 -u.d. (respectively Schnorr Σ_1^0 -u.d.)

Remark 23. In [16, Theorem 6] the result is stated for Π_1^0 sets and it is a full characterization. The passage to complements, i.e. to Σ_1^0 sets, is obvious.

3.2 Theorem 3: Characterization of randomness with Σ_1^0 -u.d.

Grouping Theorem 2 and the Effective Birkhoff Ergodic Theorem due to [18, 8, 16] and stated in Proposition 22, we obtain our third result, Theorem 3 .

Theorem 3. *Let $x \in [0, 1)$. The following conditions are equivalent.*

- i. There exists a computable sequence $(u_n)_{n \geq 1}$ of functions $[0, 1] \rightarrow \mathbb{R}$ which is computably Lipschitz, i.e. $(\ell_n)_{n \geq 1}$ -Lipschitz with $(\ell_n)_{n \geq 1}$ a computable sequence of rationals, such that the sequence $(u_n(x))_{n \geq 1}$ is Σ_1^0 -u.d.*
- ii. The sequence $(x + na)_{n \geq 1}$, for some irrational number a , is Σ_1^0 -u.d.*
- iii. The sequence $(2^n x)_{n \geq 1}$ is Σ_1^0 -u.d.*
- iv. For every computable measure-preserving and ergodic operator $T: [0, 1) \rightarrow [0, 1)$, the sequence $(T^n(x))_{n \geq 1}$ is Σ_1^0 -u.d.*
- v. x is Martin-Löf random.*

The same equivalences are valid when replacing Σ_1^0 -u.d. by Schnorr Σ_1^0 -u.d. and Martin-Löf randomness by Schnorr randomness.

Proof. (i) \Rightarrow (v) is our Theorem 2 and (v) \Rightarrow (iv) is the effective version of Birkhoff Ergodic Theorem, see Proposition 22.

For (iv) \Rightarrow (iii) consider the operator $T: [0, 1) \rightarrow [0, 1)$ such that $T(x) = 2x \bmod 1$. It is computable measure-preserving and ergodic. Observe that $T^n(x) = 2^n x \bmod 1$ hence the two sequences $(T^n(x))_{n \geq 1}$ and $(2^n x)_{n \geq 1}$ are simultaneously uniformly distributed modulo 1 or not relative to any fixed subset.

Implication (iv) \Rightarrow (ii) is similar.

Implications (iii) \Rightarrow (i) and (ii) \Rightarrow (i) are straightforward. □

Remark 24. Observe that $(x \mapsto 2^n x)_{n \geq 1}$ is a Koksma sequence of functions $[0, 1] \rightarrow \mathbb{R}$ whereas $(x \mapsto x + na)_{n \geq 1}$ is not Koksma.

Remark 25. As aforementioned in Section 1.2, the sequence $(na)_{n \geq 1}$ is u.d. when a is irrational hence, for every real x , so is $(x + na)_{n \geq 1}$. Also, $(2^n x)_{n \geq 1}$ is u.d. exactly when x is normal to base 2. But being Σ_1^0 -u.d. or Schnorr Σ_1^0 -u.d is a stronger condition.

Remark 26. We cannot replace the existential quantification in condition (i) of Theorem 3 by a universal one: consider a sequence $(u_n)_{n \in \mathbb{N}}$ such that for all n , u_n is the same Lipschitz function, then $(u_n(x))_{n \geq 1}$ is constant, hence is not u.d. and a fortiori not Σ_1^0 -u.d.

4 Proof of Theorem 1

4.1 Variations around Koksma's General Metric Theorem

Lemma 27. *We follow Vinogradov's notation $e(x)$ for $e^{2i\pi x}$. For $\mathbf{x} = (x_n)_{n \geq 1}$ a sequence of reals, we let $S_N(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^N e(x_j)$ and we denote $h\mathbf{x}$ the sequence of reals $(hx_n)_{n \in \mathbb{N}}$.*

The following are equivalent:

1. *The sequence $(x_n)_{n \geq 1}$ is u.d. mod 1.*
2. *For every $h \in \mathbb{Z} \setminus \{0\}$, $\lim_{n \rightarrow \infty} S_N(h\mathbf{x}) = 0$.*
3. *There exists a strictly increasing sequence of positive integers $(M_k)_{k \in \mathbb{N}}$ such that $M_{k+1} - M_k = o(M_k)$ and $\lim_{k \rightarrow \infty} S_{M_k}(h\mathbf{x}) = 0$ for every $h \in \mathbb{Z} \setminus \{0\}$.*

Proof. The equivalence of (1) and (2) is Weyl's Criterion ([20, Theorem 2.1 Chapter 1, p.2]). Letting $M_k = k$, implication (2) \Rightarrow (3) is trivial.

To prove its converse, observe that $|e(x)| = 1$ for all x , so that the modulus of a sum of p such exponentials is bounded by p . Thus, if $M_k \leq N < M_{k+1}$,

$$\begin{aligned} |S_N(h\mathbf{x}) - S_{M_k}(h\mathbf{x})| &= \left| \left(\frac{1}{N} - \frac{1}{M_k} \right) \sum_{\ell=1}^{M_k} e(hx_\ell) + \frac{1}{N} \sum_{\ell=M_k+1}^{M_{k+1}} e(hx_\ell) \right| \\ &\leq \frac{2(N - M_k)}{N} \\ &\leq \frac{2(M_{k+1} - M_k)}{M_{k+1}}, \end{aligned}$$

the last inequality holds because $\frac{t - M_k}{t}$ is increasing in $t \geq M_k$. Thus,

$$|S_N(h\mathbf{x})| \leq |S_{M_k}(h\mathbf{x})| + \frac{2(M_{k+1} - M_k)}{M_{k+1}}.$$

The hypothesis ensures that in the last expression both terms tend to 0. \square

Remark 28. Clearly, $M_{k+1} - M_k = o(M_k)$ implies $M_{k+1} - M_k = o(M_{k+1})$. The converse is also true: arguing by contraposition, if $M_{k+1} - M_k \geq \varepsilon M_k$ for infinitely many k 's then also $M_{k+1} - M_k \geq \frac{\varepsilon}{1 + \varepsilon} M_{k+1}$ for the same k 's.

Corollary 29. *Let x be a real and let $(M_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of positive integers such that $M_{k+1} - M_k = o(M_k)$. The following are equivalent:*

1. *For every effective Koksma sequence $\mathbf{u} = (u_n)_{n \geq 1}$, $(u_n(x))_{n \geq 1}$ is u.d.*
2. *For every effective Koksma sequence \mathbf{u} , $\lim_{k \rightarrow \infty} S_{M_k}(\mathbf{u}(x)) = 0$.*

Proof. Conditions 1 and 2 come from 1 and 3 in Lemma 27 where h is removed. This is justified because if \mathbf{u} is an effective Koksma sequence so is $h\mathbf{u}$ for $h \neq 0$. \square

Lemma 30 (Around Koksma's General Metric Theorem). *For every integer $N \geq 3$, for every $K > 0$, for every integer $h \neq 0$, for every K -Koksma sequence $\mathbf{u} = (u_n)_{n \geq 1}$, denoting $h\mathbf{u}(x)$ the sequence of reals $(hu_n(x))_{n \in \mathbb{N}}$,*

$$\int_0^1 |S_N(h\mathbf{u}(x))|^2 dx < \frac{1}{N} + \frac{8}{|h|K} \frac{\ln(3N)}{N} < \left(1 + \frac{17}{|h|K}\right) \frac{\ln(N)}{N} \quad (4)$$

Proof. The last inequality of the proof of Koksma's General Metric Theorem in [20, Theorem 4.3, Chapter 1, p.34-35] is the first above inequality. Then observe that $8 \ln(3) < 9$. \square

Inequality (4) immediately implies the inequality in the corollary below.

Corollary 31. *Let \mathbf{u} be a K -Koksma sequence. Then, for $\varepsilon > 0$,*

$$\lambda(\{x \in [0, 1] \mid |S_N(h\mathbf{u}(x))| \geq \varepsilon\}) < \frac{1}{\varepsilon^2} \left(1 + \frac{17}{|h|K}\right) \frac{\ln(N)}{N}.$$

4.2 Solovay tests for randomness

We consider the notions of Solovay randomness and total Solovay randomness, see [25, 13]. A Solovay test is a computable sequence $(V_n)_{n \geq 1}$ of computably enumerable open subsets of real numbers such that $\sum_{n \geq 1} \lambda(V_n)$ is finite. A real x passes the test if it belongs just to finitely many V_n 's. A real x is Solovay random if it passes every Solovay test. Requiring that $\sum_{n \geq 1} \lambda(V_n)$ be a computable real, we get the notions of total Solovay test and total Solovay randomness.

Proposition 32 ([22, Proposition 3.2.19] and [13, Theorem 7.1.10]). *A real passes all Solovay tests if and only if it passes all Martin-Löf tests. A real passes all total Solovay tests if and only if it passes all Schnorr tests. Thus, Solovay and Martin-Löf randomness coincide and total Solovay and Schnorr randomness also coincide.*

4.3 Proof of Theorem 1 for Martin-Löf randomness

For clarity of exposition, we first prove the weak form of Theorem 1 which states that every Martin-Löf random real is effective Koksma u.d. We argue by contradiction. Let $(u_n)_{n \geq 1}$ be an effective Koksma sequence and assume the real $x \in [0, 1]$ is such that the sequence $(u_n(x))_{n \geq 1}$ is not u.d. mod 1. We show that x is not Schnorr random by constructing a total Solovay test failed by x (cf. Proposition 32).

The argument extends in a simple way Avigad's argument in [1] using Corollary 31. For n and h , let

$$S_{N,h}(\mathbf{u}(t)) = \frac{1}{N} \sum_{j=1}^N e(h u_j(t)).$$

Letting $M_k = k^2$, we have $M_{k+1} - M_k = 2k + 1 = o(M_k)$. Applying Lemma 27, since we assumed that $(u_n(x))_{n \geq 1}$ is not u.d. mod 1, there exists an integer $h \neq 0$, such that $S_{k^2,h}(x)$ does not tend to 0 when k goes to infinity. Thus, there exists a strictly positive rational ε and infinitely many k 's such that $|S_{k^2,h}(x)| > \varepsilon$. Let

$$A_k^\varepsilon = \{t \in [0, 1] : |S_{k^2,h}(\mathbf{u}(t))| > \varepsilon\}. \quad (5)$$

Thus, x belongs to infinitely many A_k^ε 's.

Applying Corollary 31 with $N = k^2$, we get a factor 2 from $\ln(k^2)$ hence,

$$\lambda(A_k^\varepsilon) < \frac{2}{\varepsilon^2} \left(1 + \frac{17}{|h|K}\right) \frac{\ln(k)}{k^2} \quad (6)$$

As a consequence, since the series $\sum_{k \geq 1} \ln(k)/k^2$ converges so does the series $\sum_{k \geq 1} \lambda(A_k^\varepsilon)$. It remains to check that the sets A_k^ε are Σ_1^0 uniformly in k . This is immediate from the fact that the sequence $(u_j)_{j \in \mathbb{N}}$ is computable. Hence, the sequence $(A_k^\varepsilon)_{k \geq 1}$ is a Solovay test. Now, since x belongs to infinitely many A_k^ε 's, it fails this Solovay test. We conclude that x is not Martin-Löf random.

Observe that the left cut of the real $\sum_{k \geq 1} \lambda(A_k^\varepsilon)$ (the set of rational numbers less than this sum) is computably enumerable but not necessarily computable so that $(A_k^\varepsilon)_{k \geq 1}$ may not be a total Solovay test. \square

4.4 Proof of Theorem 1 for Schnorr randomness

To contradict Schnorr randomness requires some addition to the previous argument. We shall modify the open sets A_k^ε to open sets B_k^ε which are finite unions of rational intervals. This will allow to get the computability of the sum $\sum_{n \geq 1} \lambda(B_n^\varepsilon)$, ensuring that $(B_k^\varepsilon)_{k \geq 1}$ is a total Solovay test failed by x hence that x is not Schnorr random.

We shall consider both sets A_k^ε and $A_k^{\varepsilon/2}$, cf. formula (5). Let $\mathbf{u}' = (u'_n)_{n \geq 1}$ be the sequence of derivatives of the u_n 's and $\theta_k = \sup_{t \in [0,1]} |S_{k^2,h}(\mathbf{u}'(t))|$. Since the sequence of derivatives $(u'_n)_{n \geq 1}$ is computable, the same holds for the sequence $(|S_{k^2,h}(\mathbf{u}')|)_{k \geq 1}$, from which we get the computability of the sequence of reals $(\theta_k)_{k \geq 1}$. Computing rational approximations of the θ_k 's up to 1, we can define a computable function $p: \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$ such that $p(k) > \theta_k$. By the mean value theorem, we get

$$|S_{k^2,h}(\mathbf{u})(s) - S_{k^2,h}(\mathbf{u})(t)| \leq \theta_k |s - t| < p(k) |s - t|. \quad (7)$$

Define computable functions $a: \mathbb{N} \rightarrow \mathbb{N}$ and $q: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}$ such that

$$\text{for } k \geq 1, \quad 2^{-a(k)} < \varepsilon/8p(k) \quad (8)$$

$$\text{for } i \leq 2^{a(k)}, \quad q(k, i) \in \mathbb{Q} \text{ approximates } |S_{k^2,h}(\mathbf{u})(i2^{-a(k)})| \text{ up to } \varepsilon/8 \quad (9)$$

For each k , let

$$\begin{aligned} X_k &= \{i \leq 2^{a(k)} \mid q(k, i) > (3/4)\varepsilon\} \\ B_k &= [0, 1] \cap \bigcup_{i \in X_k} ((i-1)2^{-a(k)}, (i+1)2^{-a(k)}). \end{aligned}$$

If $t \in B_k$, say $t \in ((i-1)2^{-a(k)}, (i+1)2^{-a(k)})$ with $i \in X_k$, then, applying successive inequalities (7), (8), condition (9) and the definition of X_k , we get

$$\begin{aligned} |S_{k^2, h}(\mathbf{u})(t)| &\geq |S_{k^2, h}(\mathbf{u})(i2^{-a(k)})| - p(k)2^{-a(k)} \\ &\geq \left(q(k, i) - \frac{\varepsilon}{8}\right) - \frac{\varepsilon}{8} \\ &> \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

This proves inclusion $B_k \subseteq A_k^{\varepsilon/2}$.

If $t \in A_k^\varepsilon$, say $t \in [i2^{-a(k)}, (i+1)2^{-a(k)}]$, then, again applying (9), (7), recalling what means $t \in A_k^\varepsilon$, and applying inequality (8), we get

$$\begin{aligned} q(k, i) &\geq |S_{k^2, h}(\mathbf{u})(i2^{-a(k)})| - \varepsilon/8 \\ &\geq (|S_{k^2, h}(\mathbf{u})(t)| - p(k)2^{-a(k)}) - \varepsilon/8 \\ &> (\varepsilon - \varepsilon/8) - \varepsilon/8 = (3/4)\varepsilon \end{aligned}$$

hence, $i \in X_k$ and $t \in B_k$. This proves inclusion $A_k^\varepsilon \subseteq B_k$. Thus, we have $A_k^\varepsilon \subseteq B_k \subseteq A_k^{\varepsilon/2}$. The first inclusion $A_k^\varepsilon \subseteq B_k$ ensures that the real x , lying in infinitely many A_k^ε 's, also lies in infinitely many B_k 's. The second inclusion and inequality (6) (applied with $\varepsilon/2$) ensures that

$$\lambda(B_k) \leq \lambda(A_k^{\varepsilon/2}) < \alpha \frac{\ln(k)}{k^2} \quad \text{where } \alpha = \frac{8}{\varepsilon^2} \left(1 + \frac{17}{|h|K}\right) \quad (10)$$

hence, the series $\sum_{k \geq 1} \lambda(B_k)$ converges.

To see that the sum $\sum_{k \geq 1} \lambda(B_k)$ is computable, observe that, for every $L \geq 2$,

- the measure $\sum_{1 \leq k \leq L} \lambda(B_k)$ is rational and can be uniformly computed from L since B_k is a finite union of rational intervals indexed by the finite set X_k and the X_k 's are uniformly computable in k ,
- given any rational $\delta > 0$ one can compute L so that the tail $\sum_{k > L} \lambda(B_k)$ is smaller than δ . Indeed, applying inequalities (10) and the fact that $\ln(t)/t^2$ is decreasing for $t \geq 2$, we have, for $L \geq 3$,

$$\sum_{k > L} \lambda(B_k) < \alpha \sum_{k > L} \frac{\ln(k)}{k^2} < \alpha \int_L^{+\infty} \frac{\ln(t)}{t^2} dt = \alpha \frac{\ln(L) + 1}{L}.$$

This shows that the family $(B_k)_{k \geq 1}$ is a total Solovay test failed by x . □

5 Proof of Theorem 2

5.1 A needed lemma

We need the following result.

Lemma 33. *Let A be a set of real numbers and let $\text{Fr}(A) = \{\{z\} : z \in A\}$ be the set of fractional parts of elements of A .*

1. *If A is Σ_1^0 then so is $\text{Fr}(A)$ and $\lambda(\text{Fr}(A)) \leq \lambda(A)$.*
2. *If $\lambda(A)$ is computable then so is $\lambda(\text{Fr}(A))$.*

Proof. In case A is a computable interval $(n + \alpha, p + \beta)$ where n, p are integers, $n \leq p$, and $0 \leq \alpha, \beta < 1$ are computable, the result is clear since

$$\text{Fr}((n + \alpha, p + \beta)) = \begin{cases} (\alpha, \beta) & \text{if } n = p \\ [0, \beta) \cup (\alpha, 1) & \text{if } p = n + 1 \\ [0, 1) & \text{if } p \geq n + 2 \end{cases}$$

Taking a countable union we get item 1.

For item 2, observe that, up to a countable number of points, a Σ_1^0 set A is equal to a Σ_1^0 countable union $\bigcup_{k \in \mathbb{N}} I_k$ of pairwise disjoint intervals with rational endpoints. In particular, $\lambda(A) = \sum_{k \in \mathbb{N}} \lambda(I_k)$. If $\lambda(A)$ is computable then we can compute a function $n \mapsto p_n$ such that $\sum_{k > p_n} \lambda(I_k) < 2^{-n}$. Since $\lambda(\text{Fr}(I_k)) \leq \lambda(I_k)$, we see that $\lambda(\text{Fr}(\bigcup_{k > p_n} I_k))$ is a computable (in n) approximation of $\lambda(\text{Fr}(A))$ up to 2^{-n} . \square

5.2 Proof of Theorem 2 for Martin-Löf randomness

We prove the contrapositive of Theorem 2. Assume $x \in [0, 1]$ is not Martin-Löf random and let $(u_n)_{n \geq 1}$ be a computable sequence of functions $[0, 1] \rightarrow \mathbb{R}$ which is $(\ell_n)_{n \geq 1}$ -Lipschitz, where $(\ell_n)_{n \geq 1}$ is a computable sequence of rationals. We construct a Σ_1^0 set A which witnesses that the sequence $(u_n(x))_{n \geq 1}$ is not Σ_1^0 -u.d.

Define

$$P(N, X) = \frac{1}{N} \# \left\{ q : 1 \leq q \leq N, \text{Fr}(x) \in X \right\}.$$

We first give the flavor of the proof. Let $(V_n)_{n \geq 1}$ be a Martin-Löf test failed by x , that is, $x \in \bigcap_{n \geq 1} V_n$. Up to the extraction of a computable subsequence of $(V_n)_{n \geq 1}$, one can suppose that $\lambda(V_n) \leq 2^{-n-3}/\ell_n$, so that by ℓ_n -Lipschitzness we have $\lambda(u_n(V_n)) \leq 2^{-n-3}$. Consider the set $A = \bigcup_{n \in \mathbb{N}} u_n(V_n)$. Then,

$$\lambda(A) \leq \sum_{n \in \mathbb{N}} \lambda(u_n(V_n)) \leq \sum_{n \in \mathbb{N}} 2^{-n-3} = 1/4,$$

hence also

$$\lambda(\text{Fr}(A)) \leq 1/4.$$

Since x is in all V_n 's, for every n we have $\text{Fr}(u_n(x)) \in \text{Fr}(A)$ hence $P(N, \text{Fr}(A)) = 1$. This shows that the sequence $(u_n(x))_{n \geq 1}$ is not u.d. relative to the set $\text{Fr}(A)$. Unfortunately, the set $\text{Fr}(A)$ is not necessarily open.

Now we slightly modify the $u_n(V_n)$'s to get a Σ_1^0 set. Let $(I_{n,k})_{n,k \in \mathbb{N}}$ be a sequence of open intervals with rational endpoints such that $V_n = \bigcup_{k \in \mathbb{N}} I_{n,k}$ for all n and the sequences of left and right endpoints are computable. Observing that $I_{n,k} \setminus (\bigcup_{p < k} I_{n,p})$ is a finite union of pairwise disjoint intervals (possibly not open), we write $V_n = \bigcup_{k \in \mathbb{N}} J_{n,k}$ where, for each n , the sequence $(J_{n,k})_{k \in \mathbb{N}}$ consists of pairwise disjoint intervals (possibly not open) such that the sequences of left and right endpoints are computable.

Since the u_n 's are computable they are continuous and, by the mean value theorem, $u_n(J_{n,k})$ is an interval with computable endpoints $\alpha_{k,n}$ and $\beta_{k,n}$:

$$\alpha_{k,n} = \min\{u_n(t) : t \in J_{n,k}\} \quad \beta_{k,n} = \max\{u_n(t) : t \in J_{n,k}\}$$

Since u_n is ℓ_n -Lipschitz we have $\lambda(u_n(J_{n,k})) \leq \ell \lambda(J_{n,k})$. Consider the Σ_1^0 open sets

$$\omega_{n,k} = (\alpha_{n,k} - 2^{-n-k-5}, \beta_{n,k} + 2^{-n-k-5})$$

and

$$\Omega_n = \bigcup_{k \in \mathbb{N}} \omega_{n,k} \quad \text{and} \quad \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n.$$

Since $u_n(J_{n,k})$ is an interval with endpoints $\alpha_{n,k}, \beta_{n,k}$, the set Ω_n contains the $u_n(J_{n,k})$'s hence contains $u_n(V_n)$. Also,

$$\Omega_n = u_n(V_n) \cup \bigcup_{k \in \mathbb{N}} (\alpha_{n,k} - 2^{-n-k-5}, \alpha_{n,k}) \cup [\beta_{n,k}, \beta_{n,k} + 2^{-n-k-5})$$

$$\lambda(\Omega_n) \leq \lambda(u_n(V_n)) + 2 \times \sum_{k \in \mathbb{N}} 2^{-n-k-5} \leq 2^{-n-3} + 2^{-n-3} = 2^{-n-2}$$

$$\lambda(\Omega) \leq 1/2.$$

Thus, Ω is a Σ_1^0 set and by Lemma 33, $\text{Fr}(\Omega)$ is also a Σ_1^0 set. The measure of $\text{Fr}(\Omega)$ is at most $1/2$ and $P(N, \text{Fr}(\Omega)) = 1$ because for every n we have $x \in V_n$, so $u_n(x) \in u_n(V_n) \subseteq \Omega_n \subseteq \Omega$. Thus, $(u_n(x))_{n \geq 1}$ is not u.d. relative to the Σ_1^0 set $\text{Fr}(\Omega)$. \square

5.3 Proof of Theorem 2 for Schnorr randomness

In case x is not Schnorr then in the above argument we have the extra condition that the sequence $(\lambda(V_n))_{n \in \mathbb{N}}$ is computable. We show how to computably approximate $\lambda(\Omega)$ up to any given $\varepsilon > 0$. Let p be such that $2^{-p} \leq \varepsilon$.

Approximation 1. The set $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$ is approximated by $\bigcup_{n < p} \Omega_n$ up to measure 2^{-p-3} since

$$\begin{aligned} \lambda\left(\bigcup_{n \geq p} \Omega_n\right) &\leq \lambda\left(\bigcup_{n \geq p} u_n(V_n)\right) + \sum_{n \geq p, k \in \mathbb{N}} 2 \times 2^{-n-k-5} \\ &\leq \left(\sum_{n \geq p} 2^{-n-3}\right) + 2^{-p-3} \\ &< 2^{-p-1}. \end{aligned}$$

- Approximation 2.* Observe that for each n ,
- $V_n = \bigcup_{k \in \mathbb{N}} J_{n,k}$ is the disjoint union of the intervals $J_{n,k}$'s for $k \in \mathbb{N}$,
 - the sequence of their rational endpoints is computable, and
 - the measure of V_n is computable uniformly in n .

Then, we can compute q such that, for each $n < p$, the set $\bigcup_{k < q} J_{n,k}$ approximates V_n up to measure $2^{-n-p-3}/\ell_n$. We can also assume $q \geq p$. By ℓ_n -Lipschitzness, for each $n < p$, the set $\bigcup_{k < q} u_n(J_{n,k})$ approximates $u_n(V_n)$ up to measure 2^{-n-p-3} . Finally, the set $\bigcup_{n < p, k < q} u_n(J_{n,k})$ approximates $\bigcup_{n < p} u_n(J_{n,k})$ up to measure $\sum_{n < p} 2^{-n-p-3} < 2^{-p-2}$.

Approximation 3. For each n and k let

$$\rho_{n,k} = (\alpha_{n,k} - 2^{-n-k-5}, \alpha_{n,k}] \cup [\beta_{n,k}, \beta_{n,k} + 2^{-n-k-5}).$$

The set $\bigcup_{k < q} \rho_{n,k}$ approximates $\bigcup_{k \in \mathbb{N}} \rho_{n,k}$ up to measure 2^{-n-q-3} because

$$\lambda\left(\bigcup_{k \geq q} \rho_{n,k}\right) \leq \sum_{k \geq q} 2^{-n-k-4} = 2^{-n-q-3}$$

Hence $\bigcup_{n < p, k < q} \rho_{n,k}$ approximates $\bigcup_{n < p, k \in \mathbb{N}} \rho_{n,k}$ up to measure $\sum_{n < p} 2^{-n-q-3} < 2^{-q-2}$.

Conclusion. The measure of the set

$$Z = \left(\bigcup_{n < p, k < q} u_n(J_{n,k}) \right) \cup \left(\bigcup_{n < p, k < q} \rho_{n,k} \right)$$

which is made of at most $3pq$ many intervals, is computable. Using Approximations 2 and 3, we see that the set Z approximates

$$\bigcup_{n < p} \Omega_n = \left(\bigcup_{n < p, k \in \mathbb{N}} u_n(J_{n,k}) \right) \cup \left(\bigcup_{n < p, k \in \mathbb{N}} \rho_{n,k} \right)$$

up to measure $2^{-p-2} + 2^{-q-2} \leq 2^{-p-1}$ (since $q \geq p$). Using Approximation 1, the set $\bigcup_{n < p} \Omega_n$ approximates Ω up to measure 2^{-p-1} . Thus, Z approximates Ω up to measure 2^{-p} hence up to measure ε . This concludes the proof of Theorem 2. \square

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