On the Logic for Utopia

Areces, Carlos Becher, Verónica

postmast@logia.uba.ar postmaster@becher.uba.ar

Universidad de Buenos Aires, FCEyN

Departamento de Computación

Pabellón I, Ciudad Universitaria

(1428) Buenos Aires, Argentina

July, 1995

Abstract

In this study we propose the standard modal logic KDC4 as the logic governing expressions about Utopia. We define a formal construction corresponding to Utopian expressions in ordinary language that we name utopian conditionals. They possess the singular properties of admitting Strengthening of the Antecedent while possibly defeating the rule of Modus Ponens. Perhaps the most interesting aspect of this work is that, as far as the authors know, this is the first time a category of expressions in the ordinary language corresponding to these two singular properties is provided.

Keywords: theory of conditionals, modal logic, knowledge representation.

Acknowledgements The authors are indebted to Carlos Alchourrón for his valuable comments on early ideas about this paper. This research has been supported by the fellowships of both authors in the Informatic Program at the University of Buenos Aires.

1 Introduction

Usually, there are two properties that are given special attention when speaking of formalizing conditionals, Modus Ponens and Strengthening of the Antecedent. One may wonder about the different characteristics entailed by the satisfaction or not of these two crucial properties. Indeed, these characteristics that are analyzed over formal conditionals become adequacy conditions to determine which are the corresponding constructions in the ordinary language.

Much study has been devoted to delineate the boundaries of the different conditionals and the problem seems to turn more cryptic every time a new category is proposed.

However, we can speak of three kinds of ordinary conditionals for which there is a wide consensus with respect to their formalization: i. those used in *scientific proofs* are generally represented as strict conditionals (typically in an S5 context); ii. the so called *defeasible conditionals* or conditionals for defeasible inference which have received much attention in the IA community (for example the works of Lehmann [1], Boutilier [4, 5, 6] and Alchourrón [3]); iii. David Lewis' *counterfactuals* [2] modeled as variable strict conditionals over a system of spheres.

It is clear, for example, that the formalization of the conditional construction in scientific proofs satisfies both Modus Ponens (MP) and Strengthening of the Antecedent (SA), while the construction corresponding to defeasible conditionals satisfy neither. In short, we obtain the following table:

MP	**SA**	**Conditional**
Yes	Yes	Scientific Proofs
No	No	Defeasible Conditionals
Yes	No	Counterfactuals
No	Yes	?

It seems easy to find a formal construction that corresponds to the last category; namely, a conditional violating MP while validating SA. However, it is less obvious to determine whether there is a class of ordinary language conditionals corresponding to it. This is precisely what this paper addresses. We propose both, a class of expressions in the ordinary language and its formal representation, and demonstrate that they comply with our desideratum.

2 Utopian Expressions

We take an *utopian expression* as an utterance about imaginary or quixotic state of affairs. We envisage the general form of an utopian expression as a conditional assertion *if A then B*, where A and B are statements denoting ideal situations or perfect circumstances. In other words, we are thinking that in our conception of Utopia, we accept B whenever we accept A. Here is an example:

> *If wars were replaced by chess competitions then death and power would not be related any more.*

A peculiar property of these expressions is that they admit "ornamentation" of their antecedent preserving its acceptability but becoming, in this process, less informative. Namely, the more specific the antecedent the less it says about the correlation with the consequent.

However, the antecedent could be reinforced up to the extreme of ceasing to be utopian, or becoming contradictory. In such cases the utopian conditional is acceptable but meaningless, reflecting the inexistence of utopian situations where the antecedent holds. On the other side, when the antecedent is weakened to the point of becoming vacuously true, the consequent has to be a true utopian sentence, that is, it has to be valid everywhere in Utopia.

It is clear that we usually sustain an utopian expression while realizing that actuality does not behave accordingly. In other words, we can accept an utopian expression and consider its antecedent as actual, while denying its consequent. The actual world is typically a counterexample to our utterance about Utopia. In this study we embody the above interpretation into a formal language as an *utopian conditional*.

3 On the Logic for Utopia

In this section we present a logic governing the utopian expressions commented in the previous section. We establish a correspondence between an utopian expression in the ordinary language and a formal construction in the modal logic KDC4. Utopian expressions in our modal language become utopian conditionals. We take Kripke Models as the standard semantic theory for modal logic. Thus, we give semantics to sentences about Utopia by interpreting the accessibility relation between worlds as an ordering of *idealism* or *perfection*. We assume this relation to be an ordering because we insist that it should be transitive, serial and comparable. We discuss this below.

First we propose the standard modal logic KDC4 as the underlying logic for Utopia. Then, we arrive at a definition of an utopian conditional guided by the intuitions we already presented. In this section we assume familiarity with classical propositional logic as well as acquaintance with modal logics. We take L as the standard modal language. Upper-case letters A, B, C are used to denote arbitrary formulae in L. Speaking of semantic models we will denote with W the set of possible worlds, or points which are noted with Greek letters α, β, γ. We refer to A-worlds meaning that such worlds satisfy the formula A.

Definition 1 *The modal logic KDC4 is the smallest set $S \subseteq L$ such that S contains CPL and the following axioms, and is closed under the following rules of inference:*

K $\Box(A \supset B) \supset (\Box A \supset \Box B)$

D $\Box A \supset \Diamond A$

C $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B) \vee \Diamond(A \wedge \Diamond B) \vee \Diamond(\Diamond A \wedge B)$

4 $\Box A \supset \Box\Box A$

Nec *From A infer $\Box A$*

MP *From $A \supset B$ and A infer B*

US *From A infer A' where A' is a substitution instance of A*

It is well known that the logic KDC4 is sound and complete with respect to the class of transitive, connected and comparable models. A model is *transitive* if its relation R satisfies

$$\forall \alpha, \beta, \gamma((R(\alpha, \beta) \wedge R(\beta, \gamma)) \supset R(\alpha, \gamma)),$$

it is *comparable* if

$$\forall \alpha, \beta, \gamma((R(\alpha, \beta) \wedge R(\alpha, \gamma)) \supset (R(\beta, \gamma) \vee R(\gamma, \beta) \vee \beta = \gamma)),$$

and *serial* if

$$\forall \alpha \exists \beta (R(\alpha, \beta)).$$

This last condition corresponds to models with no dead ends, that is every world has access to some world which may or may not be itself.

Theorem 2 (Hughes and Cresswell 1984) *The system KDC4 is characterized by the class of transitive and comparable models, with no dead ends.*

We already mentioned that our intuition behind the accessibility relation in our models should be an ordering of worlds reflecting their "distance"[1] to Utopia. The only conditions we impose on this relation are transitivity, "weak" connectedness and no dead ends: if a world is farther from Utopia than another which is in turn farther than a third one, then the first is farther than the last; for any two accessible worlds either one is closer to Utopia than the other, or they are equally close; and, there is no world closest to Utopia. Each world considers that there is a world closer to Utopia than itself. However, there is a limiting case where a "final" world has no option but choosing itself as the least distant to Utopia. Hence, combining all requirements we obtain that every world has access to Utopia.

The resulting modal structure is a total preorder of clusters of worlds, where a cluster is a mutually accessible set of worlds[2]. A different way of describing the model for Utopia is as possibly multiple ordered sequences of worlds, each sequence ranging from Hell to Utopia. Nothing forces a sequence to be finite; that is, inferno can be a possibly infinite chain of more and more sinister worlds, while Utopia a possibly infinite chain of paradise-like states of affairs. Our actual world could be anywhere in a sequence, presumably as far from Hell as from Utopia, depending on who designs the accessibility relation; we could take it as absolute in the Universe, or as reflecting the beliefs or desires of a particular agent.

It is notorious that the strict conditionals satisfy always, disregarding the underlying accessibility relation, the rule of Strengthening of the Antecedent:

$$\text{If } \Box(A \supset B) \text{ then } \Box((A \wedge C) \supset B)$$

[1]This is not a mathematical distance but a metaphorical expresion

[2]Notice that a world in a cluster does not necessarily have access to itself.

as the simple proof below shows:

Proof: We want to prove $\Box((A \wedge C) \supset B)$, given that $\Box(A \supset B)$.
We have to see that for every model $M = \langle W, R, v\rangle$ and for every $\alpha \in W$, $M \models_\alpha \Box((A \wedge C) \supset B)$
iff for every $\beta \in W$ such that $\alpha R \beta$, $M \models_\beta (A \wedge C) \supset B$.
By hypothesis, we have that $M \models_\alpha \Box(A \supset B)$, and so $M \models_\beta A \supset B$
then by SA of the material conditional $M \models_\beta (A \wedge C) \supset B$.

It is easy to see that MP does not hold for strict conditionals for models that are not reflexive. A world can assert $\Box(A \supset B)$ while satisfying $(A \wedge \neg B)$ because it may not be accessible from itself. So strict conditionals over non-reflexive models are candidates to utopian conditionals.

Even though satisfaction of SA and not MP are mandatory to match the notion we are pursuing, they are surely incomplete with respect to Utopian expressions, in at least one way. Not every world better than ours is sufficiently utopian. The conditional must hold from a certain point in our ordering of worlds up to Utopia, but not at every point. This motivates an attempt towards a definition of an utopian conditional that we will note as $A > B$:

$$A > B \equiv_{\text{df}} \Diamond\Box(A \supset B)$$

that is, it must be possible that the strict conditional comes true.

We will now prove that our definition of $A > B$ satisfies the formal conditions we stated. It violates MP.

Proof: We have to give a model M and a world α in the model such that:
$M \models_\alpha A > B$ and $M \models_\alpha A$ but $M \not\models_\alpha B$.
We propose $M =< W, R, I >$ where $W = \{\alpha, \beta\}$, $R = \{(\alpha, \beta), (\beta, \beta)\}$,
$I(p_0) = \{\alpha, \beta\}$
$I(p_1) = \{\beta\}$
$I(p_i) = \{\} \quad \forall i > 1$
Then we have $M \models_\alpha (p_0 > p_1)$ and $M \models_\alpha p_0$ but $M \not\models_\alpha p_1$.

It satisfies SA:

Proof: Take any model M and any world α in M, and suppose that $M \models_\alpha A > B$. We want to prove that for every other formula C, $M \models_\alpha (A \wedge C) > B$.
If $M \models_\alpha A > B$ then $M \models_\alpha \Diamond\Box(A \supset B)$.

It implies that there is a world β such that $\alpha R \beta$ and $M \models_\beta \Box(A \supset B)$,
if and only if for every world γ with $\beta R \gamma$ we have $M \models_\gamma A \supset B$.
Now, by SA of the material implication, for any formula C we have $M \models_\gamma (A \wedge C) \supset B$.
As the accessibility relation is transitive, $\alpha R \beta$ and $\beta R \gamma$ implies $\alpha R \gamma$,
and then $M \models_\alpha \Diamond\Box((A \wedge C) \supset B)$.

Utopian conditionals express a fact about Utopia. It is natural to think of Utopia as a singular place, so no matter where we are, we should describe Utopia in the same way. Remarkably, our definition of an utopian conditional in KDC4 is *local*; namely, the truth conditions of an utopian conditional may vary depending on the world where the conditional is considered. However, given that KDC4 structures are disjoint ordered sequences of worlds, whenever a conditional holds in a world, it automatically holds in every world in its sequence. Therefore, an utopian conditional is *global* in a sequence. Clearly, the reason for it is that every world in a sequence has access to Utopia. We consider that a satisfactory model for Utopia should be a unique chain of worlds, as opposed to many disjoint components each reflecting distance to Utopia[3]. In such a connected model an utopian conditional becomes global: it is true in a world if and only if it is true in every world.

Given that an utopian conditional is defined as a possibility of a strict implication, an utopian conditionals inherit the paradoxes affecting strict conditionals.

$$\Box A \supset (B > A) \quad , \forall B \in L$$

and

$$\Box \neg A \supset (A > B) \quad , \forall B \in L$$

The first one asserts that if all accessible worlds satisfy A, then Utopia sanctions A. Hence, A does not need any other support to be considered utopian. Every B is irrelevant for A. The second is the dual version of the first, and can be interpreted as saying that if there are no accessible A-worlds, then A could not happen in Utopia. If we sustain A as utopian anything at all becomes conceivable.

An utopian conditional $A > B$ holds non-vacuously, that is denoting the relationship in Utopia between antecedent and consequent, when the conditional $A > \neg B$ does not hold simultaneously. Actually, when $A > B$ and $A > \neg B$ are both true, they indicate that there are no A-worlds in Utopia, or what is the same, that in every utopian state of affairs $\neg A$ holds. When both $A > B$ and $A > \neg B$ are true, so is $A > \bot$ and $\top > \neg A$. As indicated above, if A were utopian then anything could happen even inconsistencies. Since this can not be, we are left with $\neg A$ occurring in Utopia.

In the same spirit, when the conditionals $A > B$ and $\neg A > B$ are both validated, their significance reduces to assert that B holds at every world in Utopia. When none of the two conditionals $A > B$ and $A > \neg B$ hold, it follows that among the A-worlds in Utopia some are B and some are $\neg B$.

Let's analyze how the discussion above and other peculiarities are translated in terms of models.

$$M \models (A > B) \wedge (\neg A > B)$$

[3]In standard modal logic the requirement of total connectedness is not expressible, so this condition has to be specified extra-logically.

This is the case when the best worlds in M are B-worlds. In this case, the antecedent A was not necessary to arrive to B. That is, B holds on its own, no matter whether A or not A .

$$M \models (A > B) \wedge (A > \neg B)$$

This is the case when the best worlds in M are $\neg B$-worlds.

$$M \models (\neg A) > A$$

This conditional is reduced to $\Diamond\Box A$ which asserts there is an accessible point such that every successor validates A. The best worlds are A-worlds.

$$M \models \neg(A > \bot)$$

This one demands the existence of an A-world among the best worlds in M.

We can detect the sentences that holds in Utopia looking for the A such that:

$$M \models (\top > A)$$

These A have to be valid throughout all Utopia.

$$M \models (\bot > A) \quad , \forall A \in L$$

This is valid no matter the model M under consideration. Formally it says $\Diamond\Box\top$, and this is a theorem in our logic (recall that models for Utopia are serial).
Another theorem is the Identity property of the conditional.

$$M \models A > A \quad , \forall A \in L$$

Every better world has to be either A or $\neg A$, which is a trivial requirement.

Two other properties follow from our definition, Transitivity and Contraposition.

$$M \models (A > B) \wedge (B > C) \supset (A > C)$$

and

$$M \models (A > B) \supset (\neg B > \neg A)$$

These are easily proved from the fact that they hold for strict conditionals.

4 Concluding Remarks

We have proposed the standard modal logic KDC4 as the logic governing expressions about Utopia. That is, we have provided a logical calculus and semantics for utopian conditionals. These conditionals possess the singular properties of admitting Strengthening of the Antecedent while possibly defeating the rule of Modus Ponens. Perhaps, the most interesting aspect of this work is that, as far as we know, this is the first time a category of expressions in the ordinary language corresponding to these two singular properties is provided. This was a lacking category in the taxonomy of conditionals with respect to their behavior towards the two properties.

Even though the formal construction is very simple it seems to embody our intuitive concept of utopian expressions. The strength of the characterization is probably given by the quite natural properties we imposed on the accessibility relation.

This study may resemble the work on conditionals of Alchourrón, Lewis, Boutilier and Lehmann (and surely many others). Indeed, we have made use of many resources proposed by them. For instance, the analysis of properties like SA and MP as adequacy conditions, the fact that we give an intuitive reading of the accessibility relation and the differences between local and global conditionals. This paper can be seen as an exercise of putting together some of the well known pieces of conditional logics in order to characterize a set of sentences in the ordinary language, the existence of which was in doubt.

We have said nothing about embedded or iterated utopian conditionals; that is, conditionals that have an utopian conditional as its antecedent or consequent. Our definition of an utopian conditional $A > B$ admits A and B to be *any* formula in the modal language L. Undoubtedly, the logic handles embedded conditionals. What is less clear is whether there are utopian expressions in the ordinary language corresponding to them.

As Carlos Alchourrón[4] has indicated to us, an interesting avenue remains to be explored. He suggested a comparison between our logic and deontic systems, as they are governed by modal logic KD4. This comparison can result in some useful insight in the relation between an order of things as *we want them to be* and the order that *must be*.

References

[1] D. Lehmann. 1989. What does a Conditional Knowledge Base Entail? In *Proceedings of First International Conference of Principles of Knowledge Representation and Reasoning*, 212-222, Toronto.

[2] D. Lewis. 1973. *Counterfactuals.* Blackwell, Oxford.

[3] C. Alchourrón. 1993. Defeasible, Strict and Counterfactual Conditionals. Nonmonotonic Inference Relations. In *Conditionals and Artificial Intelligence* Edited by Crocco, Fariñas del Cerro, Herzig. Toulouse, France.

[4] C. Boutilier. 1992. Conditional Logics for belief revision and Default Reasoning. PhD. Thesis. Technical Report KRR-TR-92-1. University of Toronto, Toronto, Ontario.

[5] C. Boutilier. 1994. Conditional Logics of Normality: A Modal Approach. *Artificial Intelligence*, 68:87-154.

[4] Personal Communication

[6] C. Boutilier. 1994. Unifying Default Reasoning and Belief Revision in a Modal Framework. *Artificial intelligence*, 68: 33-85.

[7] G. E. Hughes and M. J. Cresswell. 1984. *A Companion to Modal Logic*. Methuen. London.