

On a question of Mendès France on normal numbers

by

VERÓNICA BECHER (Buenos Aires) and
MANFRED G. MADRITSCH (Vandœuvre-lès-Nancy)

1. Introduction and statement of results. In this note we solve a problem posed by Michel Mendès France asking for an instance of a real number x such that both x and $1/x$ are simply normal to a given integer base b . The problem appeared in the literature in 2008 in [19] and it was presented to us by Gerhard Larcher.

The continued fraction representation of a positive number and its reciprocal are identical except for a shift one place left or right depending on whether the number is less than 1 or greater than 1, respectively. That is, the numbers represented by $[a_0; a_1, a_2, \dots]$ and $[0; a_0, a_1, \dots]$ are reciprocals. This fact allows us to prove the following extension of the problem of Mendès France.

THEOREM 1. *There is a number x such that both x and its reciprocal $1/x$ are continued fraction normal and absolutely normal. Moreover, they are both computable, and the first n digits of their continued fraction expansions can be computed in $\mathcal{O}(n^4)$ mathematical operations.*

We construct x and $1/x$ by defining incrementally their continued fraction expansions. To ensure that both x and $1/x$ are continued fraction normal and absolutely normal we follow the work by Becher and Yuhjtman [3]. The challenge in the present paper is to handle simultaneously *two* constructions, one for x and one for $1/x$. These constructions work by defining successive refinements of appropriate subintervals to achieve, in the limit, in both cases, continued fraction normality and simple normality to all integer bases. At each step the choice of digits for the two constructions is done without invoking the digits chosen at previous steps. This explains why the computation is very efficient.

2020 *Mathematics Subject Classification*: Primary 11K16, 11J70.

Key words and phrases: normal numbers, continued fraction normal, computable numbers.

Received 13 August 2021; revised 14 January 2022.

Published online *.

2. Two types of normality. We follow the standard notation in this area. For a detailed account of normal numbers see [12, 8, 6, 2], for symbolic dynamics see [13, 7] and for a combination of both see [14].

As usual we write $\mathbb{N} = \{1, 2, 3, \dots\}$. For a finite set S , we denote by $\#S$ its cardinality. Similarly for an infinite set S of real numbers, $|S|$ denotes its Lebesgue measure. We use Landau's notation for the asymptotic behaviour of functions: $g(x) = \mathcal{O}(f(x))$ if there exist constants x_0 and c such that for every $x \geq x_0$, $|g(x)| < c \cdot |f(x)|$. We write \log for the logarithm to base e .

2.1. Continued fraction normality. For a real number $x \in [0, 1]$, the continued fraction expansion of x is the integer part $a_0 = \lfloor x \rfloor$ together with a sequence of positive integers a_1, a_2, \dots such that

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \frac{1}{\ddots}}}}}$$

we write $x = [a_0; a_1, a_2, \dots]$ for short. This expansion can be seen as an infinite word over the alphabet \mathbb{N} . Since normality is an asymptotic property of the digits, we drop the integer part in what follows and write $[a_1, a_2, \dots]$ instead of $[a_0; a_1, a_2, \dots]$.

A way of obtaining the continued fraction expansion is applying the Gauss map $T: [0, 1] \rightarrow [0, 1]$ defined by

$$T(x) = \begin{cases} 1/x - \lfloor 1/x \rfloor & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $x = [a_1, a_2, \dots]$ then $T^n(x) = [a_{n+1}, a_{n+2}, \dots]$ and for every $n \geq 1$, $a_n = \lfloor 1/T^{n-1}(x) \rfloor$. In other words, the Gauss map corresponds to the left shift in the associated symbolic dynamical system over the alphabet \mathbb{N} .

The map T has an invariant ergodic measure, the Gauss measure μ , which is absolutely continuous with respect to Lebesgue measure (see Dajani and Kraaikamp [7]). In particular, for every Lebesgue measurable set A , we have

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx.$$

An interval I in the unit interval is a cylinder set of order n with respect to the continued fraction expansion, or *cf-ary of order n* , if there is a finite continued fraction

$$[a_1, \dots, a_n]$$

such that I is the set of all numbers whose first n digits of their continued fraction expansion are a_1, \dots, a_n . Thus,

$$\begin{aligned} I_{[a_1, \dots, a_n]} &= ([a_1, \dots, a_n], [a_1, \dots, a_n + 1]), \quad \text{or} \\ I_{[a_1, \dots, a_n]} &= ([a_1, \dots, a_n + 1], [a_1, \dots, a_n]) \end{aligned}$$

depending on whether n is even or odd, respectively. The set of cf -ary intervals of order n forms a partition of the unit interval into infinitely many parts of different lengths.

A real number $x = [a_1, a_2, \dots]$ is *continued fraction normal*, or *cf-normal* for short, if every word of positive integers occurs in its continued fraction expansion with the asymptotic frequency determined by the Gauss measure. In other words, x is generic for μ , i.e. for every positive integer k and for every word $v_1 \dots v_k$ in \mathbb{N}^k , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \#\{j : 1 \leq j \leq n, a_j = v_1, \dots, a_{j+k-1} = v_k\} = \mu(I_{[v_1, \dots, v_k]}).$$

In order to get a feeling for cf -normality we provide some remarks. Quadratic irrationals are never cf -normal, because their expansions are periodic. However, nothing is known about algebraic numbers of higher degree. The number $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$ is not cf -normal because it is the concatenation of the pattern $(1m1)$, for all even m in increasing order, and no other odd digit except 1 occurs in the expansion. Nothing else is known about cf -normality of other transcendental constants. By Birkhoff's Ergodic Theorem [4] almost every real in the unit interval is cf -normal and there are several constructions of cf -normal numbers.

2.2. Normality to integer bases. For an integer $b \geq 2$ called the *base* we denote by $\mathcal{N}_b = \{0, \dots, b-1\}$ the corresponding set of digits. Every positive integer n has a unique representation of the form

$$n = a_\ell b^\ell + \dots + a_1 b + a_0$$

with $a_i \in \mathcal{N}_b$ for $0 \leq i \leq \ell$. This representation can be extended to real numbers x in $[0, 1]$ by

$$x = \sum_{i=1}^{\infty} a_i b^{-i}$$

with $a_i \in \mathcal{N}_b$ for $i \geq 1$ and $a_i \neq b-1$ infinitely often. The latter ensures that every rational number has a unique representation. As in the case of continued fraction expansions, there exists a map of the unit interval that describes the dynamical aspect of the b -ary expansion, namely $S_b : [0, 1] \rightarrow [0, 1]$ defined by

$$S_b(x) = bx - [bx].$$

An interval $I \subset [0, 1]$ is a cylinder set of order n with respect to the b -ary expansion, or *b-ary of order n* for short, if there is a finite word $d_1 \dots d_n$ over \mathcal{N}_b such that I is the set of real numbers whose first n digits of their b -ary expansion are d_1, \dots, d_n . The set of b -ary intervals of order n forms a partition of the unit interval into finitely many parts of equal length (in contrast to the infinitely many parts of different lengths in the case of the continued fraction expansion).

A real $x = a_1 b^{-1} + a_2 b^{-2} + \dots \in [0, 1[$ is *simply normal* with respect to base b if every digit occurs in the b -ary expansion of x with the same asymptotic frequency $1/b$. That is, for each $v \in \mathcal{N}_b$,

$$\lim_{n \rightarrow \infty} \frac{\#\{1 \leq j \leq n: a_j = v\}}{n} = \frac{1}{b}.$$

Normality to base b is simple normality to bases b, b^2, b^3, \dots , for all powers of b (this definition of normality is equivalent to Borel's original definition [5], as proved by Pillai in 1940 [6, Theorem 4.2]). *Absolute normality* is normality to every integer base $b \geq 2$; hence, simple normality to every integer base $b \geq 2$. Borel showed that almost all real numbers (with respect to Lebesgue measure) are absolutely normal. This also follows from Birkhoff's Ergodic Theorem [4], since Lebesgue measure is ergodic with respect to the map S_b (see Dajani and Kraaikamp [7]).

3. Lemmas. We use the classical notion of discrepancy, but not on arbitrary intervals. For continued fraction expansions we consider the classical discrepancy restricted to cf-ary intervals. Similarly, for b -ary expansions we consider the classical discrepancy restricted to b -ary intervals.

DEFINITION (Discrepancy for continued fraction representation). For a finite word $\mathbf{v} = v_1 \dots v_k$ over the alphabet \mathbb{N} we define the *discrepancy* of $x = [a_1, a_2, \dots]$ with respect to \mathbf{v} in the first n positions of the continued fraction expansion of x by

$$D_{\mathbf{v}, n}^{\text{cf-ary}}(x) = \left| \frac{1}{n} \#\{j : 1 \leq j \leq n, a_j = v_1, \dots, a_{j+k-1} = v_k\} - \mu(I_{[v_1, \dots, v_k]}) \right|.$$

With some notational abuse, if \mathbf{w} is a sequence of digits in \mathbb{N} , then we write $D_{\mathbf{v}, n}^{\text{cf-ary}}(\mathbf{w})$ for $D_{\mathbf{v}, n}^{\text{cf-ary}}(x)$, where x is the real number whose continued fraction expansion is \mathbf{w} .

Clearly, a real number x is continued fraction normal if and only if for every positive integer k , and for every $\mathbf{v} \in \mathbb{N}^k$,

$$\lim_{n \rightarrow \infty} D_{\mathbf{v}, n}^{\text{cf-ary}}(x) = 0.$$

DEFINITION (Discrepancy for integer base representation). For a real $x = \sum_{j \geq 1} a_j b^{-j}$ we define the *discrepancy* of the digit $v \in \mathcal{N}_b$ among the

first n digits of the b -ary expansion of x by

$$D_{v,n}^{b\text{-ary}}(x) = \left| \frac{1}{n} \#\{1 \leq j \leq n: a_j = v\} - \frac{1}{b} \right|,$$

and we set

$$D_n^{b\text{-ary}}(x) = \max_{v \in \mathcal{N}_b} D_{v,n}^{b\text{-ary}}(x).$$

Again with abuse of notation, if \mathbf{w} is a sequence of digits in \mathcal{N}_b , then we write $D_n^{b\text{-ary}}(\mathbf{w})$ for $D_n^{b\text{-ary}}(x)$, where x is the real number with b -ary expansion \mathbf{w} .

Clearly, x is simply normal to base b if and only if

$$\lim_{n \rightarrow \infty} D_n^{b\text{-ary}}(x) = 0.$$

3.1. Length of continued fraction intervals. We start by considering the length of different continued fraction intervals. For $x = [a_1, a_2, \dots]$ we recursively define the functions $p_n(x)$ and $q_n(x)$, called the *convergents* of x , as follows. We set

$$p_{-1}(x) = q_0(x) = 1 \quad \text{and} \quad p_0(x) = q_{-1}(x) = 0,$$

and recursively for $n \geq 1$,

$$p_n(x) = a_n p_{n-1}(x) + p_{n-2}(x), \quad q_n(x) = a_n q_{n-1}(x) + q_{n-2}(x).$$

On the one hand, for irrational $x = [a_1, a_2, \dots]$, $p_n(x)/q_n(x)$ is the n th approximant to x and converges to x as $n \rightarrow \infty$. On the other hand, for rational $x = [a_1, \dots, a_n]$, we have $x = p_n(x)/q_n(x)$ and we write $q(x)$ for $q_n(x)$. Observe that for every x , $(p_n(x))_{n \geq 1}$ and $(q_n(x))_{n \geq 1}$ are increasing. Furthermore, the length of a cf-ary interval is

$$|I_{[a_1, \dots, a_n]}| = \frac{1}{q_n(q_n + q_{n-1})}.$$

LEMMA 1. For $n \in \mathbb{N}$ and $a_i \in \mathbb{N}$ for $2 \leq i \leq n$ we have

$$|I_{[0; a_2, \dots, a_n]}|/4 \leq |I_{[0; 1, a_2, \dots, a_n]}| \leq |I_{[0; a_2, \dots, a_n]}|.$$

Proof. This is a special case of [3, Lemma 3]. ■

The distribution of $\log q_n$ obeys in the limit a Gaussian law. This was first proved by Ibragimov [10]. Then Philipp [18, Satz 3] obtained an error term of $\mathcal{O}(n^{-1/5})$, which was later improved by Mischyavichyus [16] to $\mathcal{O}(n^{-1/2} \log n)$. Morita [17, Theorem 8.1] obtained the optimal error term $\mathcal{O}(n^{-1/2})$; a different proof was given by Vallée [20, Théorème 9]. This yields the following lemma that ensures that there are many disjoint large cf-ary subintervals of relative order n inside any given interval I .

We write L for Lévy's constant $\pi^2/(12 \log 2)$.

LEMMA 2 ([3, Lemma 5]). There are positive constants K, C and a positive integer N_1 such that for any cf-ary interval I and any integer $n \geq N_1$,

the Lebesgue measure of the union of the cf -ary subintervals J of I of relative order n such that

$$\frac{|I|}{4}e^{-2nL-2C} \leq |J| \leq 2|I|e^{-2nL+2C}$$

is greater than $K|I|/\sqrt{n}$.

3.2. Length of continued fraction intervals with large discrepancy. The following result on large deviations is essentially from Kifer, Peres and Weiss [11, Corollary 3.2] but conditioned on the first r terms.

LEMMA 3 ([3, Lemma 6]). *Let $I_{[a_1, \dots, a_r]}$ be a cf -ary interval of order r , and let $\mathbf{v} = v_1 \dots v_k$ be a word of length k over \mathbb{N} . Then for every positive real δ and every positive integer n ,*

$$|\{x \in I_{[a_1, \dots, a_r]} : D_{\mathbf{v}, n}^{\text{cf-ary}}(T^r x) > \delta\}| \leq 6Me^{-\frac{\delta^2 n}{2M}} |I_{[a_1, \dots, a_r]}|,$$

where

$$M = M(\delta, k) = \lceil k - \log(\delta^2 / (2 \log 2)) \rceil.$$

Recall that T is the Gauss map.

3.3. Discrepancy for continued fraction expansions. If \mathbf{w} and \mathbf{u} are words, we write \mathbf{wu} for their concatenation. The following lemma describes the change of discrepancy under concatenation.

LEMMA 4 ([3, Lemma 7]). *Let $\mathbf{v} = v_1 \dots v_k$, $\mathbf{w} = a_1 \dots a_{n+k-1}$ and $\mathbf{u} = b_1 \dots b_{s+k-1}$ be finite words over \mathbb{N} . Furthermore, let $0 < \epsilon < 1$.*

- (1) *If $D_{\mathbf{v}, n}^{\text{cf-ary}}(\mathbf{w}) < \epsilon$ and $D_{\mathbf{v}, s}^{\text{cf-ary}}(\mathbf{u}) < \epsilon - (k-1)/s$ then $D_{\mathbf{v}, n+s}^{\text{cf-ary}}(\mathbf{wu}) < \epsilon$.*
- (2) *If $D_{\mathbf{v}, n}^{\text{cf-ary}}(\mathbf{w}) < \epsilon$ and $s/n < \epsilon$ then*
 - (a) *for every $1 \leq \ell \leq s$, $D_{\mathbf{v}, n+\ell}^{\text{cf-ary}}(\mathbf{wu}) < 2\epsilon$,*
 - (b) *$D_{\mathbf{v}, n+s}^{\text{cf-ary}}(\mathbf{uw}) < 2\epsilon$.*

3.4. Length of b -ary subintervals. For any integer $b \geq 2$, every b -ary interval I of order k is of the form

$$I = \left(\frac{a}{b^k}, \frac{a+1}{b^k} \right)$$

for some positive integer k and an integer a with $0 \leq a < b^k$. We write $\text{order}_b(I) = k$. If I is a union of two consecutive b -ary intervals of the same order,

$$I = \left(\frac{a}{b^k}, \frac{a+2}{b^k} \right),$$

we also write $\text{order}_b(I) = k$. We drop the index b if the base is clear. The following is a trivial fact about lengths of b -ary subintervals.

LEMMA 5. *Let $b \geq 2$ and $m \in \mathbb{N}$. Every interval I of length less than b^{-m} is contained in a b -ary interval of order m or in a union of two such intervals.*

3.5. Number of b -ary words with large discrepancy. In the construction we use the following classical bound for the number of blocks of a given length having larger discrepancy than a given value [9, Theorem 148].

LEMMA 6 (Bernstein inequality). *Let I_{a_1, \dots, a_r} be a b -ary interval. For every positive integer n and every real δ such that $6/n \leq \delta \leq 1/b$ we have*

$$|\{x \in I_{a_1, \dots, a_r} : D_n^{b\text{-ary}}(S_b^r x) > \delta\}| \leq 2b^{n+1} e^{-b\delta^2 n/6} |I_{a_1, \dots, a_r}|.$$

Recall $S_b x = bx - \lfloor bx \rfloor$.

3.6. Discrepancy for b -ary expansions. Since there are only finitely many digits in the b -ary expansion, the bounds for the discrepancy are easier in that case.

LEMMA 7 ([1, Lemma 3.1]). *Let \mathbf{u} and \mathbf{v} be blocks in base b and let $\epsilon > 0$.*

- (1) *If $D_{|\mathbf{u}|}^{b\text{-ary}}(\mathbf{u}) < \epsilon$ and $D_{|\mathbf{v}|}^{b\text{-ary}}(\mathbf{v}) < \epsilon$, then $D_{|\mathbf{uv}|}^{b\text{-ary}}(\mathbf{uv}) < \epsilon$.*
- (2) *If $D_{|\mathbf{v}|}^{b\text{-ary}}(\mathbf{v}) < \epsilon$ and $|\mathbf{u}|/|\mathbf{v}| < \epsilon$, then*
 - (a) *for every $\ell \leq |\mathbf{u}|$, $D_{|\mathbf{v}|+\ell}^{b\text{-ary}}(\mathbf{vu}) < 2\epsilon$,*
 - (b) *$D_{|\mathbf{v}|+|\mathbf{u}|}^{b\text{-ary}}(\mathbf{uv}) < 2\epsilon$.*

4. Proof of Theorem 1. We split the proof of Theorem 1 into three parts. First we give the construction of x and $y := 1/x - \lfloor 1/x \rfloor$. Secondly, we prove that x and $1/x$ are both continued fraction normal and absolutely normal. Finally, we show that both numbers are computable and we give an upper bound of the computational complexity.

4.1. The construction of x and $y := 1/x - \lfloor 1/x \rfloor$. For each, we follow the construction of a continued fraction normal and absolutely normal number of Becher and Yuhjtman [3], which in turn is based on the work [1] on absolutely normal numbers. For a similar construction for a normal number with respect to all Pisot bases see Madritsch, Scheerer and Tichy [15].

The constructions of x and y run simultaneously. To control continued fraction normality we use cf-ary intervals and to control normality in each integer base b we use b -ary intervals. For this we define a t -brick and a refinement of a t -brick.

DEFINITION (t -brick). For an integer $t \geq 2$, a t -brick is a tuple $(\sigma_{\text{cf}}, \sigma_2, \dots, \sigma_t)$ as follows:

- the interval σ_{cf} is cf-ary,

- for each $b = 2, \dots, t$, σ_b is either a b -ary interval or the union of two consecutive b -ary intervals of the same order,
- for each $b = 2, \dots, t$,

$$\sigma_{\text{cf}} \subseteq \sigma_b, \quad |\sigma_{\text{cf}}| \geq \frac{|\sigma_b|}{4 \cdot 16e^{4C}b},$$

where C is the constant determined in Lemma 2.

DEFINITION (Refinement of a t -brick). A t -brick $\vec{\sigma} = (\sigma_{\text{cf}}, \sigma_2, \dots, \sigma_t)$ is *refined* by a t' -brick $\vec{\tau} = (\tau_{\text{cf}}, \tau_2, \dots, \tau_{t'})$ if

- $t' = t$ or $t' = t + 1$,
- $\tau_{\text{cf}} \subseteq \sigma_{\text{cf}}$,
- for $b = 2, \dots, t$, $\tau_b \subseteq \sigma_b$.

The refinement is said to have *discrepancy* less than ϵ if

- the new cf-word \mathbf{w} corresponding to the inclusion $\tau_{\text{cf}} \subseteq \sigma_{\text{cf}}$ is such that for every word \mathbf{v} of t digits $\leq t$, $D_{\mathbf{v}, |\mathbf{w}|}^{\text{cf-ary}}(\mathbf{w}) < \epsilon - (t - 1)/|\mathbf{w}|$,
- for each $b = 2, \dots, t$ the new word \mathbf{w} in base b corresponding to the inclusion $\tau_b \subseteq \sigma_b$ has simple discrepancy $D_{|\mathbf{w}|}^{b\text{-ary}}(\mathbf{w})$ less than ϵ .

Notice that if $t' > t$ the definition of a refinement of a t -brick gives no condition on $\tau_{t'}$.

Iteratively we define two sequences of refinements of t -bricks

$$\vec{\sigma}_1, \vec{\sigma}_2, \dots, \quad \vec{\Sigma}_1, \vec{\Sigma}_2, \dots$$

for non-decreasing values of t . The intersection of all the intervals in the first sequence defines the number x , whereas the intersection of all the intervals in the second sequence defines y .

Before starting with the actual construction we provide a lemma ensuring that the sequence of refinements of t -bricks exists.

4.1.1. The refinement lemma

LEMMA 8. Let $t \geq 2$ be an integer, let ϵ be a positive real less than $1/t$ and let t' be t or $t + 1$. There is an integer function $n_0 = n_0(t, \epsilon)$ such that for every $n \geq n_0$ there are positive integers ℓ_1, \dots, ℓ_n satisfying the following:

- (1) for any pair of t -bricks $(\sigma_{\text{cf}}, \sigma_2, \dots, \sigma_t)$ and $(\Sigma_{\text{cf}}, \Sigma_2, \dots, \Sigma_t)$ there are refinements $(\tau_{\text{cf}}, \tau_2, \dots, \tau_{t'})$ and $(\mathcal{T}_{\text{cf}}, \mathcal{T}_2, \dots, \mathcal{T}_{t'})$, both with discrepancy less than ϵ ,
- (2) if $\sigma_{\text{cf}} = [1, a_2, \dots, a_N]$ and $\Sigma_{\text{cf}} = [a_2, \dots, a_N]$ then

$$\tau_{\text{cf}} = [1, a_2, \dots, a_N, \ell_1, \dots, \ell_n] \quad \text{and} \quad \mathcal{T}_{\text{cf}} = [a_2, \dots, a_N, \ell_1, \dots, \ell_n].$$

Proof. First, we assume that $t' = t$.

Length of τ_{cf} and \mathcal{T}_{cf} . For a cf-interval α and a positive integer n consider the finite set $\mathcal{I}_n(\alpha)$ of cf-ary subintervals A of α of relative order n such that

$$(1) \quad \frac{1}{4}e^{-2nL-2C} \leq |A|/|\alpha| \leq 2e^{-2nL+2C}.$$

Let K, C, N_1 be the constants provided by Lemma 2. Then, if $n \geq N_1$,

$$\frac{|\bigcup_{A \in \mathcal{I}_n(\alpha)} A|}{|\alpha|} \geq \frac{K}{\sqrt{n}}.$$

For each n , consider the sets $\mathcal{I}_n(\sigma_{\text{cf}})$ and $\mathcal{I}_n(\Sigma_{\text{cf}})$. Note that by our choice of σ_{cf} and Σ_{cf} these sets have the same cardinality and there is a one-to-one correspondence between the elements by adding the digit 1 to those in $\mathcal{I}_n(\Sigma_{\text{cf}})$. At the end of the proof we will determine a value n_0 for n and we will choose τ_{cf} in $\mathcal{I}_n(\sigma_{\text{cf}})$ and \mathcal{T}_{cf} in $\mathcal{I}_n(\Sigma_{\text{cf}})$ such that

$$\begin{aligned} \frac{1}{4}e^{-2n_0L-2C}|\sigma_{\text{cf}}| &\leq |\tau_{\text{cf}}| \leq 2e^{-2n_0L+2C}|\sigma_{\text{cf}}|, \\ \frac{1}{4}e^{-2n_0L-2C}|\sigma_{\text{cf}}| &\leq |\mathcal{T}_{\text{cf}}| \leq 2e^{-2n_0L+2C}|\Sigma_{\text{cf}}|. \end{aligned}$$

And by Lemma 1 we have

$$|\mathcal{T}_{\text{cf}}|/4 \leq |\tau_{\text{cf}}| \leq |\mathcal{T}_{\text{cf}}|.$$

Length of τ_b and \mathcal{T}_b . For each $b = 2, \dots, t$ we set

$$m_b = \text{order}_b(\mathcal{T}_b) = \text{order}_b(\tau_b).$$

By the definition of a t -brick we have

$$(2) \quad |\mathcal{T}_{\text{cf}}| \leq b^{-m_b}.$$

We choose m_b as the largest integer such that

$$2e^{-2nL+2C}|\Sigma_{\text{cf}}| \leq b^{-m_b}.$$

Thus

$$b^{-m_b-1} < 2e^{-2nL+2C}|\Sigma_{\text{cf}}|.$$

Using the left inequality in (1) we obtain

$$(3) \quad b^{-m_b-1} < 2e^{-2nL+2C}|\Sigma_{\text{cf}}| = 8e^{4C}\frac{1}{4}e^{-2nL-2C}|\Sigma_{\text{cf}}| \leq 8e^{4C}|I|.$$

For every $i \in \mathcal{I}_n(\sigma_{\text{cf}})$ and for the corresponding $I \in \mathcal{I}_n(\Sigma_{\text{cf}})$ we have

$$|I|/4 \leq |i| \leq |I|$$

and from (3) we obtain

$$(4) \quad b^{-m_b-1} < 4 \cdot 8e^{4C}|i|.$$

Then, for each $i \in \mathcal{I}_n(\sigma_{\text{cf}})$ and for the corresponding $I \in \mathcal{I}_n(\Sigma_{\text{cf}})$ we determine τ_b^i and \mathcal{T}_b^I as the b -ary intervals of order m_b or the union of two consecutive b -ary intervals of order m_b that contain i and I respectively (Lemma 5) with the same choice for τ_b^i and \mathcal{T}_b^I . Thus, either $|\tau_b^i| = |\mathcal{T}_b^I| = b^{-m_b}$ or

$|\tau_b^i| = |\mathcal{T}_b^I| = 2b^{-m_b}$. Putting together (2)–(4) we obtain

$$(5) \quad \frac{1}{2 \cdot 8e^{4C}b} \leq \frac{|I|}{|\mathcal{T}_b^I|} \leq \frac{4|i|}{|\tau_b^i|}.$$

We give bounds on the number of digits we add in the b -ary expansion. For this we write

$$n_b = \text{order}(\mathcal{T}_b) - \text{order}(\Sigma_b).$$

Since

$$|\Sigma_{\text{cf}}| \leq |\Sigma_b| \leq |\Sigma_{\text{cf}}|2 \cdot 8e^{4C}b$$

and by Lemma 5, Σ_b consists of one or two b -ary intervals,

$$\text{order}(\Sigma_b) = -\log_b(|\Sigma_b|) \quad \text{or} \quad \text{order}(\Sigma_b) = -\log_b(|\Sigma_b|/2),$$

we have

$$\log_b(|\Sigma_{\text{cf}}|/2) \leq -\text{order}(\Sigma_b) \leq \log_b(|\Sigma_{\text{cf}}|8e^{4C}b).$$

And since

$$2e^{-2nL+2C}|\Sigma_{\text{cf}}| \leq b^{-m_b} \leq b \cdot 2e^{-2nL+2C}|\Sigma_{\text{cf}}|$$

we have

$$\log_b(2e^{-2nL+2C}|\Sigma_{\text{cf}}|) \leq -\text{order}(\mathcal{T}_b) = -m_b \leq \log_b(b \cdot 2e^{-2nL+2C}|\Sigma_{\text{cf}}|).$$

For the number n_b of digits we add to the b -ary expansion, we obtain

$$\begin{aligned} 2nL \log_b e - \log_b(4be^{2C}) &\leq \text{order}(\mathcal{T}_b) - \text{order}(\Sigma_b) = n_b \\ &\leq 2nL \log_b e + \log_b(4e^{2C}b). \end{aligned}$$

Thus,

$$(6) \quad 2n \frac{L}{\log b} - \frac{2C}{\log b} - 3 \leq n_b \leq 2n \frac{L}{\log b} + \frac{2C}{\log b} + 3.$$

Bad zones. We must pick one interval i in $\mathcal{I}_n(\sigma_{\text{cf}})$ and one interval I in $\mathcal{I}_n(\Sigma_{\text{cf}})$ in a zone of low discrepancy. This is possible because the measure of the zones of large discrepancy decreases at an exponential rate in n , while the measure of $\mathcal{I}_n(\sigma_{\text{cf}})$ and $\mathcal{I}_n(\Sigma_{\text{cf}})$ decreases only as K/\sqrt{n} . For each n let

$$B_{b,\sigma_b,m_b,\epsilon}^0 \quad \text{and} \quad B_{b,\Sigma_b,m_b,\epsilon}^0$$

be the sets of reals in the b -ary subintervals of σ_b and Σ_b respectively of order m_b with b -discrepancy greater than ϵ . And let

$$B_{b,\sigma_b,m_b,\epsilon} \quad \text{and} \quad B_{b,\Sigma_b,m_b,\epsilon}$$

be the union of $B_{b,\sigma_b,m_b,\epsilon}^0$ and $B_{b,\Sigma_b,m_b,\epsilon}^0$ with those numbers lying in a b -ary interval of the same order that is a neighbour to one in $B_{b,\sigma_b,m_b,\epsilon}^0$ and $B_{b,\Sigma_b,m_b,\epsilon}^0$ respectively.

Recall that m_b is the order of τ_b and \mathcal{T}_b , which we reach by adding n_b digits to the intervals σ_b and Σ_b , respectively. To define τ_b we need to add

n_b digits avoiding b -discrepancy greater than ϵ . Thus, using the conditions $6/n_b \leq \epsilon \leq 1/b$, Lemma 6 provides the estimate

$$\frac{|B_{b,\sigma_b,m_b,\epsilon}|}{|\sigma_b|} = \frac{|B_{b,\Sigma_b,m_b,\epsilon}|}{|\Sigma_b|} \leq 6be^{-b\epsilon^2 n_b/6}.$$

Notice that the factor 6 comes from considering the b -ary intervals in B which are those in B^0 together with their neighbour b -ary intervals to the left and to the right. By (5),

$$|\sigma_b| \leq 4|\sigma_{\text{cf}}| \cdot 2 \cdot 8e^{4C}b, \quad |\Sigma_b| \leq |\Sigma_{\text{cf}}| \cdot 2 \cdot 8e^{4C}b$$

and from (6) we know

$$n_b \geq 2n \frac{L}{\log b} - \frac{2C}{\log b} - 3.$$

We obtain

$$\frac{|B_{b,\sigma_b,m_b,\epsilon}|}{|\sigma_{\text{cf}}|} \leq \frac{4|B_{b,\Sigma_b,m_b,\epsilon}|}{|\Sigma_{\text{cf}}|} \leq A(b)e^{-b\epsilon^2 Ln/(3 \log b)},$$

where

$$A(b) = 384e^{4Cb^2}e^{b\epsilon^2(\frac{C}{3 \log b} + \frac{1}{2})}.$$

Consider the bad zones with respect to the continued fraction expansion. For each n , let

$$\tilde{B}_{t,\Sigma_{\text{cf}},n,\epsilon} \quad \text{and} \quad \tilde{B}_{t,\sigma_{\text{cf}},n,\epsilon}$$

be the set of reals x in the respective cf-ary subintervals of Σ_{cf} and σ_{cf} of relative order n such that for some word of length t of digits less than or equal to t the cf-discrepancy of x is greater than $\epsilon - (t-1)/n$.

With the condition $2(t-1)/\epsilon \leq n$, it suffices to consider cf-discrepancies greater than $\epsilon/2$. Then Lemma 3 gives the estimate

$$\frac{|\tilde{B}_{t,\Sigma_{\text{cf}},n,\epsilon}|}{|\Sigma_{\text{cf}}|} = \frac{|\tilde{B}_{t,\sigma_{\text{cf}},n,\epsilon}|}{|\sigma_{\text{cf}}|} \leq t^t 6Me^{-\frac{(\epsilon/2)^2 n}{2M}},$$

where

$$M = \left\lceil t - \log \left(\frac{(\epsilon/2)^2}{2 \log 2} \right) \right\rceil.$$

Find n_0 large enough. We choose n_0 such that the measure of the union of the bad zones of σ_{cf} and Σ_{cf} as well as the bad zones of σ_b and Σ_b for $b = 2, \dots, t$ is small enough so that we can find an interval in $\mathcal{I}_n(\sigma_{\text{cf}})$ and an interval in $\mathcal{I}_n(\Sigma_{\text{cf}})$ outside the bad zones and defined by appending the same n_0 cf-digits to the cf-expansion of Σ_{cf} and σ_{cf} .

We let n_0 be the least integer n such that for each $b = 2, \dots, t$,

$$A(b)e^{-b\epsilon^2 Ln/(3 \log b)} < \frac{1}{8} \frac{K}{t\sqrt{n}}, \quad 6Mt^t e^{-\frac{(\epsilon/2)^2 n}{2M}} < \frac{1}{8} \frac{K}{t\sqrt{n}},$$

where the factor $1/8$ ensures that

- (1) less than $1/8$ of the measure of $\mathcal{I}_n(\sigma_{\text{cf}})$ is covered with bad zones with respect to the continued fraction expansion of σ_{cf} ;
- (2) less than $1/8$ of the measure of $\mathcal{I}_n(\sigma_{\text{cf}})$ is covered with the projection of the bad zones with respect to the corresponding continued fraction expansion of Σ_{cf} ;
- (3) less than $1/8$ of the measure of $\mathcal{I}_n(\sigma_{\text{cf}})$ is covered with bad zones with respect to the b -ary expansion inside σ_{cf} and inside Σ_{cf} ;
- (4) at least $5/8$ of the measure of $\mathcal{I}_n(\sigma_{\text{cf}})$ is free of bad zones;
- (5) the above four points also hold on interchanging σ_{cf} with Σ_{cf} .

In turn, this ensures the existence of n digits ℓ_1, \dots, ℓ_n such that

- if $\sigma_{\text{cf}} = [1, a_2, \dots, a_N]$ and $\Sigma_{\text{cf}} = [a_2, \dots, a_N]$ then

$$\tau_{\text{cf}} = [1, a_2, \dots, a_N, \ell_1, \dots, \ell_n] \quad \text{and} \quad \mathcal{T}_{\text{cf}} = [a_2, \dots, a_N, \ell_1, \dots, \ell_n],$$

- $\tau_{\text{cf}} \in \mathcal{I}_n(\sigma_{\text{cf}})$, $\mathcal{T}_{\text{cf}} \in \mathcal{I}_n(\Sigma_{\text{cf}})$,
- τ_{cf} and \mathcal{T}_{cf} are not in bad zones.

So, we need to find solutions to

$$\sqrt{n} e^{-rn} \leq \gamma$$

for certain values of r and γ . Since $x < e^{x/2}$ for every $x > 0$, we have

$$\sqrt{n} e^{-rn/2} \leq \frac{1}{r} r n e^{-rn/2} < \frac{1}{r} e^{rn/2 - rn/2} = \frac{1}{r}.$$

Thus, we need n such that

$$e^{-rn/2} \leq \gamma r$$

for each of the needed values r and γ . Hence, n has to be as large as

$$-(2/r) \log(\gamma r)$$

for each of the needed values r and γ . Letting

$$r^{(1)} = \epsilon^2/(8M) \quad \text{and} \quad \gamma^{(1)} = K/(6Mt^{t+1})$$

and for $b = 2, \dots, t$,

$$r^{(b)} = b\epsilon^2 L/(3 \log b) \quad \text{and} \quad \gamma^{(b)} = K/(t A(b)),$$

taking

$$n = \max \left\{ -(2/r^{(b)}) \log(\gamma^{(b)} r^{(b)}) : 2 \leq b \leq t \right\} \cup \left\{ \frac{6}{\epsilon}, \frac{2(t-1)}{\epsilon}, N_1 \right\}$$

(recall that N_1 is the constant already fixed at the beginning of this proof, provided by Lemma 2) completes the proof in case $t' = t$.

The case $t' = t+1$ follows easily by taking first t -bricks $\vec{\tau} = (\tau_{\text{cf}}, \tau_2, \dots, \tau_t)$ and $\vec{\mathcal{T}} = (\mathcal{T}_{\text{cf}}, \mathcal{T}_2, \dots, \mathcal{T}_t)$ refining $\vec{\sigma}$ and $\vec{\Sigma}$ respectively with discrepancy less than ϵ . Since the refinement requires no discrepancy condition on τ_{t+1} or \mathcal{T}_{t+1} , we only need to take $(t+1)$ -ary intervals τ_{t+1} and \mathcal{T}_{t+1} of order m_{t+1} ,

or a union of two consecutive such intervals so that $|\tau_{t+1}| = |\mathcal{T}_{t+1}|$, $\tau_{\text{cf}} \subseteq \tau_{t+1}$, $\mathcal{T}_{\text{cf}} \subseteq \mathcal{T}_{t+1}$ where m_{t+1} is maximal such that $|\mathcal{T}_{\text{cf}}| \leq (t+1)^{-m_{t+1}}$. Applying Lemma 5 and using $|\tau_{\text{cf}}| \leq |\mathcal{T}_{\text{cf}}|$ we obtain

$$|\tau_{\text{cf}}| \geq \frac{|\tau_{t+1}|}{2(t+1)} \quad \text{and} \quad |\mathcal{T}_{\text{cf}}| \geq \frac{|\mathcal{T}_{t+1}|}{2(t+1)}.$$

This ensures that $\vec{\mathcal{T}} = (\mathcal{T}_{\text{cf}}, \mathcal{T}_2, \dots, \mathcal{T}_{t+1})$ and $\vec{\tau} = (\tau_{\text{cf}}, \tau_2, \dots, \tau_{t+1})$ are $(t+1)$ -bricks. ■

4.1.2. The iterative construction. For simplicity, we fix x in the interval $(1/2, 1)$ so that the integer part of x is 0, the first digit in the continued fraction expansion of x is 1, and $1/x$ is in $(1, 2)$. Since $\lfloor 1/x \rfloor = 1$ and $y := x - \lfloor 1/x \rfloor$, it follows that $y \in (0, 1)$.

We define a sequence of nested intervals for x and another sequence of nested intervals for y . The construction is done step-by-step. At each step, the subintervals for x and y to be in are chosen independently of the subintervals chosen in previous steps. These are the largest that avoid the bad zones, which means that they are the largest that avoid cf-ary and b -ary intervals corresponding to words with large discrepancy.

We set the largest integer base t , the discrepancy value ϵ and the relative order n of the new cf-ary interval as functions of s . In particular, we define, for every positive integer s ,

$$t(s) = \max(2, \lfloor \sqrt[5]{\log s} \rfloor), \quad \epsilon(s) = 1/t(s).$$

Clearly $t(s)$ is non-decreasing unbounded and $\epsilon(s)$ is non-increasing and goes to zero. Consider the function $n_0(\epsilon(s), t(s))$ given by Lemma 8 below and notice that $n_0(\epsilon(s), t(s)) = \mathcal{O}(t(s)^4 \log(t(s)))$. Let n_{start} be the minimum positive integer such that for every positive s ,

$$\lfloor \log s \rfloor + n_{\text{start}} \geq n_0(\epsilon(s), t(s)),$$

and define

$$n_0(s) = \lfloor \log s \rfloor + n_{\text{start}}.$$

The following is unchanged in all steps s of the construction for $\vec{\sigma}_s = (\sigma_{\text{cf}}, \sigma_2, \dots, \sigma_{t(s)})$ and $\vec{\Sigma}_s = (\Sigma_{\text{cf}}, \Sigma_2, \dots, \Sigma_{t(s)})$:

$$|\Sigma_{\text{cf}}|/4 \leq |\sigma_{\text{cf}}| \leq |\Sigma_{\text{cf}}|,$$

and for each $b = 2, \dots, t(s)$,

$$|\sigma_b| = |\Sigma_b|,$$

$$\sigma_{\text{cf}} \subseteq \sigma_b, \quad |\sigma_{\text{cf}}| \geq |\sigma_b|/(4 \cdot 16e^{4C}b),$$

$$\Sigma_{\text{cf}} \subseteq \Sigma_b, \quad |\Sigma_{\text{cf}}| \geq |\Sigma_b|/(16e^{4C}b).$$

Initial step, $s = 1$:

$$\begin{aligned}\vec{\sigma}_1 &= (\sigma_{\text{cf}}, \sigma_2) \quad \text{with } \sigma_2 = \sigma_{\text{cf}} = (1/2, 1) = I_{[0;1]}, \\ \vec{\Sigma}_1 &= (\Sigma_{\text{cf}}, \Sigma_2) \quad \text{with } \Sigma_2 = \Sigma_{\text{cf}} = (0, 1) = I_{[0;]}.\end{aligned}$$

Iterative step, $s > 1$: Assume we already have two bricks

$$\vec{\sigma}_{s-1} = (\sigma_{\text{cf}}, \sigma_2, \dots, \sigma_{t(s-1)}) \quad \text{and} \quad \vec{\Sigma}_{s-1} = (\Sigma_{\text{cf}}, \Sigma_2, \dots, \Sigma_{t(s-1)}).$$

We choose $\vec{\sigma}_s = (\tau_{\text{cf}}, \tau_2, \dots, \tau_{t(s)})$ and $\vec{\Sigma}_s = (\mathcal{T}_{\text{cf}}, \mathcal{T}_2, \dots, \mathcal{T}_{t(s)})$ such that if $\sigma_{\text{cf}} = [a_1, \dots, a_N]$ and $\Sigma_{\text{cf}} = [a_2, \dots, a_N]$ then

$$\begin{aligned}\tau_{\text{cf}} &= [a_2, \dots, a_N, a_{N+1}, \dots, a_{N+n_0(s)}], \\ \mathcal{T}_{\text{cf}} &= [a_1, \dots, a_N, a_{N+1}, \dots, a_{N+n_0(s)}]\end{aligned}$$

are the leftmost cf-subintervals of σ_{cf} and Σ_{cf} of relative order $n_0(s)$ ensuring that $\vec{\sigma}_s$ refines $\vec{\sigma}_{s-1}$ and $\vec{\Sigma}_s$ refines $\vec{\Sigma}_{s-1}$, both with discrepancy less than $\epsilon(s)$.

4.2. Correctness of the construction. The existence of the sequences $\vec{\sigma}_1, \vec{\sigma}_2, \dots$ and $\vec{\Sigma}_1, \vec{\Sigma}_2, \dots$ is guaranteed by Lemma 8. Let x and y be defined by the intersection of all the intervals in the respective sequences.

4.2.1. x and $1/x$ are continued fraction normal. The construction ensures that, removing the first digit in the continued fraction expansion of x , the continued fractions of x and y are identical. Since $y = 1/x - \lfloor 1/x \rfloor = 1/x - 1$, to show that x and $1/x$ are continued fraction normal it suffices to show that x and y are continued fraction normal.

Let $\mathbf{v} = v_1 \dots v_m$ be a word of length m over \mathbb{N} and let $\tilde{\epsilon} > 0$. Choose s_0 so that $m \leq t(s_0)$, $\max\{v_1, \dots, v_m\} \leq t(s_0)$ and $\epsilon(s_0) \leq \tilde{\epsilon}/4$. At each step s after s_0 , the continued fraction expansions of x and y are constructed by appending a word u_s such that $|u_s| = n_0(s)$ and

$$D_{\mathbf{v}, |u_s|}^{\text{cf-ary}}(u_s) < \epsilon(s) - \frac{t(s-1) - 1}{|u_s|} < \epsilon(s) - \frac{m-1}{|u_s|}.$$

By Lemma 4(1) applied several times, for every $s \geq s_0$ we obtain

$$D_{\mathbf{v}, |u_{s_0} \dots u_s|}^{\text{cf-ary}}(u_{s_0} u_{s_0+1} \dots u_s) < \epsilon(s_0).$$

By Lemma 4(2b) there is s_1 sufficiently large such that for every $s \geq s_1$,

$$D_{\mathbf{v}, |u_1 \dots u_s|}^{\text{cf-ary}}(u_1 \dots u_s) < 2\epsilon(s_0).$$

Since $n_0(s)$ grows logarithmically, the inequality

$$n_0(s) \leq 2\epsilon(s_0) \sum_{j=1}^{s-1} n_0(j)$$

holds from a certain point on. Hence, by Lemma 4(2a), for every s sufficiently large and every ℓ such that $|u_1 \dots u_{s-1}| < \ell \leq |u_1 \dots u_s|$,

$$D_{\mathbf{v}, \ell}^{\text{cf-ary}}(u_1 \dots u_s) < 4\epsilon(s_0) < \tilde{\epsilon}.$$

It follows that x and y are continued fraction normal.

4.2.2. x and $1/x$ are absolutely normal. Absolute normality follows by showing simple normality to all integer bases ≥ 2 . We prove that x is simply normal to all integer bases $b \geq 2$; the case of y is similar. Since $1/x = [1/x] + y = 1 + y$, we conclude that $1/x$ is also simply normal to all integer bases $b \geq 2$.

Fix $b \geq 2$ and let $\tilde{\epsilon} > 0$. We choose s_0 such that $t(s_0) \geq b$ and $\epsilon(s_0) \leq \tilde{\epsilon}/4$. At each step s after s_0 the expansion of x in base b was constructed by appending blocks u_s such that $D_{|u_s|}^{b\text{-ary}}(u_s) < \epsilon(s_0)$. Thus, by Lemma 7(1), for any $s > s_0$,

$$D_{|u_{s_0} \dots u_s|}^{b\text{-ary}}(u_{s_0} \dots u_s) < \epsilon(s_0).$$

Applying Lemma 7(2a), we obtain s_1 such that for any $s > s_1$,

$$D_{|u_1 \dots u_s|}^{b\text{-ary}}(u_1 \dots u_s) < 2\epsilon(s_0).$$

Let $n_b(j)$ be the relative order of the b -interval of $\vec{\sigma}_j$ with respect to the b -interval of $\vec{\sigma}_{j-1}$. The inequalities

$$2n_0(j) \frac{L}{\log b} - \frac{2C}{\log b} - 3 \leq n_b(j) \leq 2n_0(j) \frac{L}{\log b} + \frac{2C}{\log b} + 3,$$

provided by (6) in the proof of Lemma 8, tell us that $n_b(j)$ grows logarithmically. Then, for s sufficiently large,

$$n_b(s) \leq 2\epsilon(s_0) \sum_{j=1}^{s-1} n_b(j).$$

By Lemma 7(2b) we conclude that for s sufficiently large and $|u_1 \dots u_{s-1}| \leq \ell \leq |u_1 \dots u_s|$,

$$D_{\ell}^{b\text{-ary}}(u_1 \dots u_s) < 4\epsilon(s_0) < \tilde{\epsilon}.$$

Thus x is simply normal to base b , for every $b \geq 2$.

4.3. x and $1/x$ are efficiently computable. A real number is computable if, for some integer $b \geq 2$, there is an algorithm that produces the consecutive digits of its b -ary expansion. In addition to Lévy's constant $L = \pi^2/(12 \log 2)$, our construction of x and y depends on three constants, K , C and N_1 indicated in Lemma 2. Since these three constants can be taken to be integers (and they need not be minimal), there is an algorithm that, for any given integer $b \geq 2$, produces the b -ary expansion of x and $1/x$. Therefore, x and $1/x$ are computable.

The computational complexity of our construction is exactly that in [3, Section 3.3]. We follow that analysis verbatim. We count mathematical operations and do not count how many elementary operations are implied by each of them, meaning that we neglect the computational cost of performing arithmetical operations with arbitrary precision.

At the beginning of step s the current t -bricks are

$$\vec{\sigma}_{s-1} = (\sigma_{\text{cf}}, \sigma_2, \dots, \sigma_{t(s-1)}) \quad \text{and} \quad \vec{\Sigma}_{s-1} = (\Sigma_{\text{cf}}, \sigma_2, \dots, \sigma_{t(s-1)}).$$

We assume that at step s the construction has direct access to the left endpoint of each of σ_{cf} , Σ_{cf} , σ_b and Σ_b for $b = 2, \dots, t(s-1)$, as well as their lengths. The construction divides σ_{cf} and Σ_{cf} into

$$\lfloor 4e^{2n(s)L+2c} \rfloor + 1$$

equal intervals. In order to find the demanded t -bricks inside σ_{cf} and inside Σ_{cf} we have to inspect, in the worst case, all the candidate endpoints. Since $n(s) = \lfloor \log s \rfloor + n_{\text{start}}$, the total number T of candidate endpoints in each of σ_{cf} and Σ_{cf} is

$$\lfloor 4e^{2(\lfloor \log s \rfloor + n_{\text{start}})L+2c} \rfloor.$$

Thus, the number T of endpoints is $\mathcal{O}(s^{2L})$. We need to compute $n(s)$ digits of the continued fraction expansion of each candidate endpoint in σ_{cf} and in Σ_{cf} and determine if its discrepancy is suitable and if the interval length is suitable. Define

$$N(s) = \sum_{i=1}^s n(i).$$

Let $z_{\text{cf}} = [1, a_1, \dots, a_{N(s-1)}]$ and $Z_{\text{cf}} = [a_1, \dots, a_{N(s-1)}]$ be the left endpoints of σ_{cf} and Σ_{cf} respectively. Furthermore let z_b and Z_b be the left endpoints of σ_b and Σ_b respectively, for $b = 2, \dots, t(s-1)$. For $j = 0, \dots, T-1$, let f_j be the positive integers such that $f_j = \lfloor \sigma_{\text{cf}} \rfloor j / T$.

We write each candidate endpoint e_j and E_j , for $j = 0, \dots, T-1$, as

$$e_j = z_{\text{cf}} + f_j \quad \text{and} \quad E_j = Z_{\text{cf}} + f_j.$$

Then the continued fraction expansion of e_j is $[1, a_1, \dots, a_{N(s-1)}]$ concatenated with the continued fraction expansion of f_j . Similarly, the continued fraction expansion of E_j is $[a_1, \dots, a_{N(s-1)}]$ concatenated with the continued fraction expansion of f_j . We only need $n(s)$ digits of the continued fraction expansion of f_j , which we may obtain by running the Euclidean algorithm on pairs of integers u, v such that $u/v = f_j$ for $n(s)$ iterations. This requires $\mathcal{O}(n(s))$ mathematical operations.

Let $\mathcal{I}_{\text{cf}}(\sigma_{\text{cf}})$ and $\mathcal{I}_{\text{cf}}(\Sigma_{\text{cf}})$ be the cf-ary subintervals of σ_{cf} and Σ_{cf} respectively of relative order $n(s)$. For each of these, the computation of its length requires computing the convergents $q_{N(s-1)+1}, \dots, q_{N(s-1)+n(s)}$. For each interval, checking that the length is suitable requires $\mathcal{O}(n(s))$ mathematical operations.

Now we write each endpoint e_j and E_j , for $j = 0, \dots, T - 1$, as

$$e_j = (z_{\text{cf}} - z_b) + z_b + f_j \quad \text{and} \quad E_j = (Z_{\text{cf}} - Z_b) + Z_b + f_j.$$

Then the b -ary expansion of e_j consists of the b -ary expansion of z_b concatenated with the b -ary expansion of $(z_{\text{cf}} - z_b) + f_j$, and similarly for the b -ary expansion of E_j . By the proof of Lemma 8, for each base b , we just need n_b digits of this expansion and n_b is $\mathcal{O}(n(s))$. The conversion of the rational values $(z_{\text{cf}} - z_b) + f_j$ and $(Z_{\text{cf}} - Z_b) + f_j$ to base b can be done in a constant number of operations.

Finally, we need to check if the discrepancy of each of the t blocks witnessed by e_j and E_j is less than $\epsilon(s)$. This can be done by a number of comparisons that is linear in the length of the block plus a constant number of operations, hence in $\mathcal{O}(n(s))$ operations.

We conclude that at step s in the worst case the number of required mathematical operations to choose $\vec{\sigma}_s$ and $\vec{\Sigma}_s$ is

$$\mathcal{O}(T(n(s) + n(s) + t(s) \cdot \text{constant} + t(s)n(s))).$$

Since $T = \mathcal{O}(s^{2L})$, $n(s) = \mathcal{O}(\log(s))$, and $t(s) = \mathcal{O}(\log^{1/5}(s))$, the total number of mathematical operations at step s is

$$\mathcal{O}(s^{2L} \log^{6/5}(s)).$$

After the first k steps the number of digits of the continued fraction expansions of x and y obtained is respectively $N(k) + 1$ and $N(k)$, which are both greater than k . The number of mathematical operations performed by the construction is of the order of

$$\sum_{s=1}^k s^{2L} \log^{6/5}(s) \leq k^{2L+1} \log^{6/5}(k),$$

and this last expression is $\mathcal{O}(k^4)$. This completes the proof of Theorem 1.

Acknowledgements. The first author is supported by grant STIC-Amsud 20-STIC-06 and PICT-2018-02315 funded by Agencia Nacional de Promoción Científica y Tecnológica de Argentina. The second author is supported by project ANR-18-CE40-0018 funded by the French National Research Agency. The problem was brought to the authors' attention at the workshop on "Discrepancy Theory and Applications – Part 2", CIRM 2641, February 2021, organized by Manfred Madritsch, Joël Rivat and Robert Tichy.

References

- [1] V. Becher, P. A. Heiber, and T. Slaman, *A polynomial-time algorithm for computing absolutely normal numbers*, Information and Computation 232 (2013), 1–9.
- [2] V. Becher and O. Carton, *Normal numbers and computer science*, in: Sequences, Groups, and Number Theory, Birkhäuser, 2018, 233–269.

- [3] V. Becher and S. Yuhjtman, *On absolutely normal and continued fraction normal numbers*, Int. Math. Res. Notices 2019, 6136–6161.
- [4] G. D. Birkhoff, *Proof of the Ergodic Theorem*, Proc. Nat. Acad. Sci. USA 17 (1931), 656–660.
- [5] É. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Suppl. Rend. Circ. Mat. Palermo 27 (1909), 247–271.
- [6] Y. Bugeaud, *Distribution Modulo One and Diophantine Approximation*, Cambridge Tracts in Math. 193, Cambridge Univ. Press, Cambridge, 2012.
- [7] K. Dajani and C. Kraaikamp, *Ergodic Theory of Numbers*, Carus Math. Monogr. 29, Math. Assoc. Amer., Washington, DC, 2002.
- [8] M. Drmota and R. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Math. 1651, Springer, 1997.
- [9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford Univ. Press, Oxford, 2008.
- [10] I. Ibragimov, *A theorem from the metric theory of continued fractions*, Vestnik Leningrad. Univ. 16 (1961), no. 1, 13–24 (in Russian).
- [11] Y. Kifer, Y. Peres and B. Weiss, *A dimension gap for continued fractions with independent digits*, Israel J. Math. 124 (2001), 61–76.
- [12] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Dover Publ., New York, 2006.
- [13] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995.
- [14] M. Madritsch, *Normal numbers and symbolic dynamics*, in: Sequences, Groups, and Number Theory, Birkhäuser, 2018, 271–329.
- [15] M. G. Madritsch, A.-M. Scheerer and R. F. Tichy, *Computable absolutely Pisot normal numbers*, Acta Arith. 184 (2018), 7–29.
- [16] G. A. Misevičius, *Estimate of the remainder in the limit theorem for the denominators of continued fractions*, Litovsk. Mat. Sb. 21 (1987), no. 3, 63–74 (in Russian).
- [17] T. Morita, *Local limit theorem and distribution of periodic orbits of Lasota–Yorke transformations with infinite Markov partitions*, J. Math. Soc. Japan 46 (1997), 309–343; corrections in 47 (1997), 191–192.
- [18] W. Philipp, *Ein zentraler Grenzwertsatz mit Anwendungen auf die Zahlentheorie*, Z. Wahrsch. Verw. Gebiete 8 (1967), 185–203.
- [19] T. Rivoal, *On the bits counting function of real numbers*, J. Austral. Math. Soc. 85 (2008), 95–111.
- [20] B. Vallée, *Opérateurs de Ruelle–Mayer généralisés et analyse des algorithmes d’Euclide et de Gauss*, Acta Arith. 81 (1997), 101–144.

Verónica Becher
 Departamento de Computación
 Facultad de Ciencias Exactas y Naturales
 Universidad de Buenos Aires & ICC CONICET
 Pabellón I, Ciudad Universitaria
 1428 Buenos Aires, Argentina
 E-mail: vbecher@dc.uba.ar

Manfred G. Madritsch
 Université de Lorraine
 Institut Élie Cartan de Lorraine
 UMR 7502
 F-54506 Vandœuvre-lès-Nancy, France
 and
 CNRS, Institut Élie Cartan de Lorraine
 UMR 7502
 F-54506 Vandœuvre-lès-Nancy, France
 E-mail: manfred.madritsch@univ-lorraine.fr

Abstract (will appear on the journal's web site only)

In 2008 or earlier, Michel Mendès France asked for an instance of a real number x such that both x and $1/x$ are simply normal to a given integer base b . We give a positive answer to this question by constructing a number x such that both x and its reciprocal $1/x$ are continued fraction normal as well as normal to all integer bases greater than or equal to 2. Moreover, x and $1/x$ are computable, the first n digits of their continued fraction expansion can be obtained in $\mathcal{O}(n^4)$ mathematical operations.