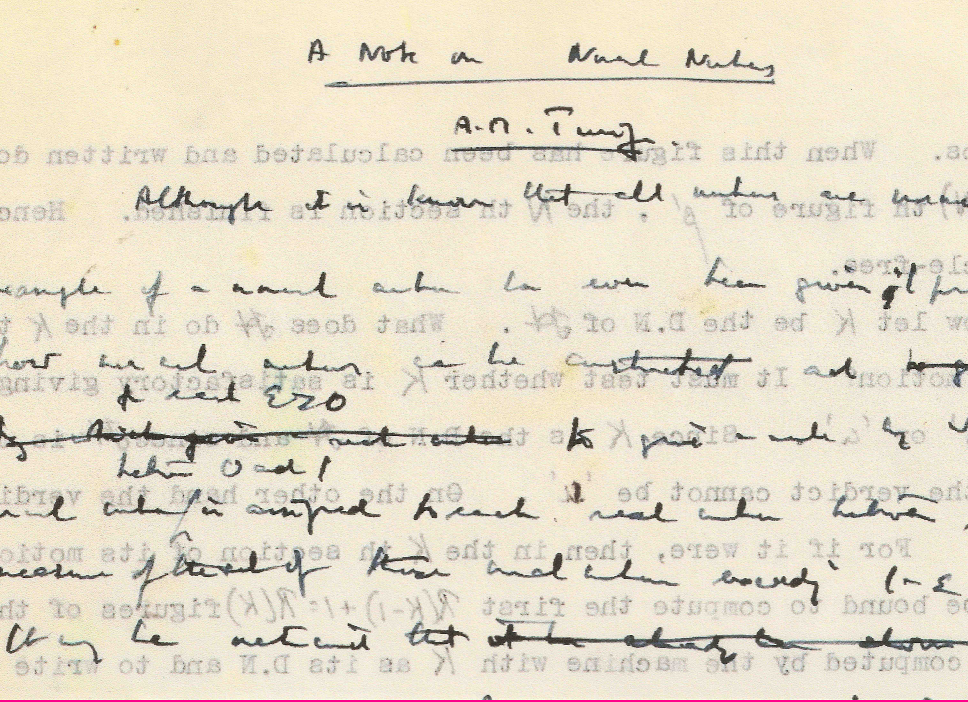
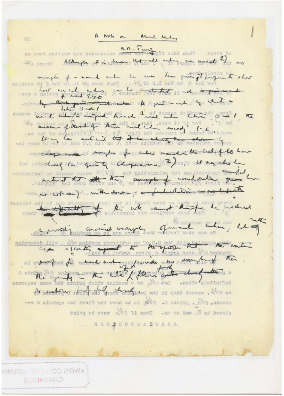
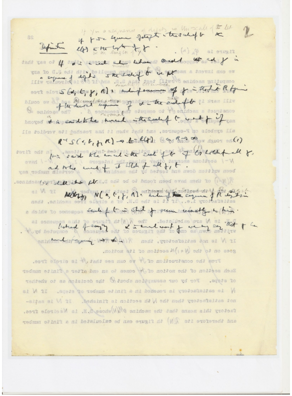
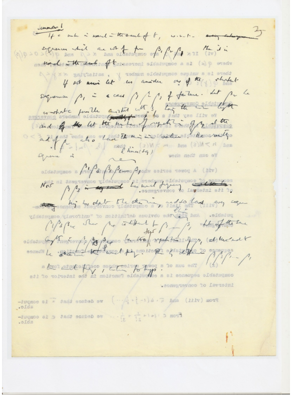
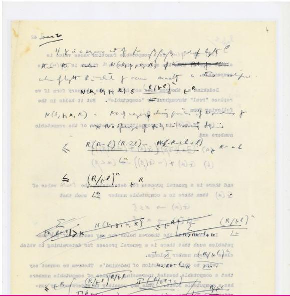
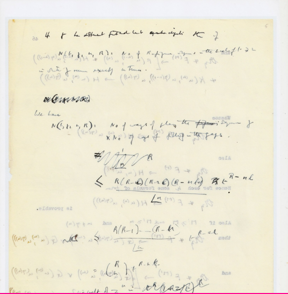
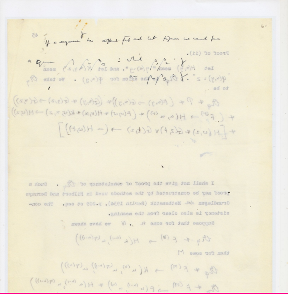
Turing's Normal Numbers

Verónica Becher

Universidad de Buenos Aires

CMU math logic seminar, November 29, 2022


A Note on Normal Numbers
A. M. Turing

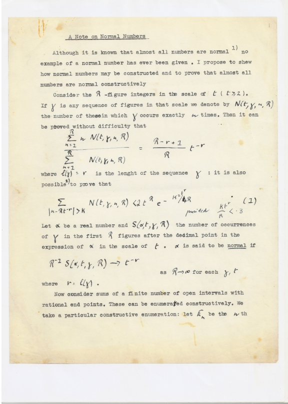
A Note on Normal Numbers

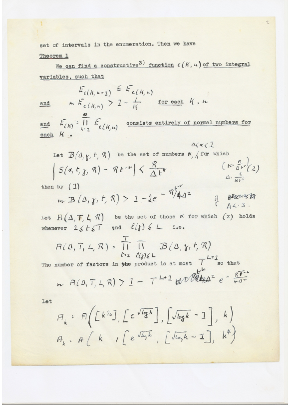
Although it is known that almost all numbers are normal [1)] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

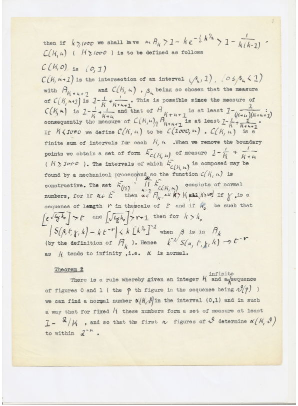
Consider the R -figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t, \gamma,$ the number of thesein which γ occurs exactly n times. Then it can be proved without difficulty that

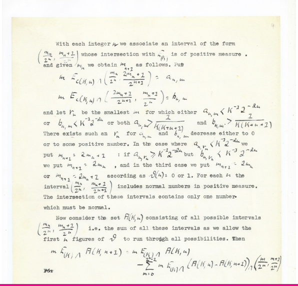
$$\frac{\sum_{n=1}^{R} n\, N(t, \gamma, n, R)}{\sum_{n=1}^{R} N(t, \gamma, n, R)} = \frac{R - r + 1}{R} t^{-r}$$

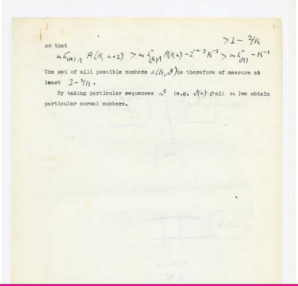
where $l(\gamma) = r$ is the lenght of the sequence γ : it is also











A Note on Normal Numbers, Alan M. Turing

Written presumably in 1936.

Unpublished until 1992, when included in the Collected Works edited by J.L.Britton. An editorial note says that the proof of the second theorem is inadequate and speculates that the theorem could be false.

A Note on Normal Numbers, Alan M. Turing

Written presumably in 1936.

Unpublished until 1992, when included in the Collected Works edited by J.L.Britton. An editorial note says that the proof of the second theorem is inadequate and speculates that the theorem could be false.

Reconstructed, corrected and completed in 2007
Becher, Figueira, Picchi, *Theoretical Computer Science* 377, 126-138.

Normality, a form of randomness

Defined by Émile Borel in 1909, 1922:

Normality, a form of randomness

Defined by Émile Borel in 1909, 1922:

A real number is *normal to a given integer base* if in its expansion in that base every block of digits of the same length occurs with the same limit frequency.

For instance, if a number is normal to base 2, each of the digits '0' and '1' occur in the limit, half of the times; each of the blocks '00', '01', '10' and '11' occur one fourth of the times, and so on.

Normality, a form of randomness

Defined by Émile Borel in 1909, 1922:

A real number is *normal to a given integer base* if in its expansion in that base every block of digits of the same length occurs with the same limit frequency.

A real number that is normal to every integer base is called *absolutely normal*, or just *normal*.

Counterexamples

0.1010010001000010000010000... not normal to base 2.

Counterexamples

0.101001000100001000001000000... not normal to base 2.

0.10101010101010101010101010101... not normal to base 2.

Counterexamples

0.101001000100001000001000000... not normal to base 2.

0.10101010101010101010101010101... not normal to base 2.

Rationals are not normal (they have a periodic expansion in each base).

Existence

Theorem (Borel 1909)

The set of normal numbers in the unit interval has Lebesgue measure 1.

Existence

Theorem (Borel 1909)

The set of normal numbers in the unit interval has Lebesgue measure 1.

Borel asked for an explicit example.

A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the R -figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of thesein which γ occurs exactly n times. Then it can be ptoved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\, t^{-r}$$

where $\ell(\gamma) = r$ is the lenght of the sequence γ : it is also possible [2] to prove that

$$\sum \quad N(t,\gamma,n,R) < 2t^{R} e^{-K^2 t^r / 4R} \qquad (1)$$

Turing's Note on Normal Numbers

Turing's Theorem 1

Borel's theorem on the measure of normal numbers, constructively.

Turing's Theorem 2

An algorithm to construct normal numbers.

Turing's First Page of the Handwritten Manuscript

His own appraisal of his work.

Turing's Theorem 1

Theorem 1

We can find a constructive[3] function $c(K, n)$ of two integral variables, such that

$$E_{c(K,n+1)} \subseteq E_{c(K,n)}$$

and $\quad m E_{c(K,n)} > 1 - \frac{1}{K}$ for each K, n

and $\quad E_{(K)} = \prod_{n=1}^{\infty} E_{c(K,n)}$ consists entirely of normal numbers for each K.

Turing's Theorem 1

There is a computable function $c(k, n)$ of two integer numbers with values in finite sets of pairs of rational numbers,

$$c(k, n) = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$$

with the following property. Let $E_{c(k,n)}$ be set of real numbers in the union of the open intervals whose endpoints are the pairs in $c(k, n)$,

$$E_{c(k,n)} = \bigcup_{(a,b)\in c(k,n)} (a, b)$$

Then,

Turing's Theorem 1

There is a computable function $c(k, n)$ of two integer numbers with values in finite sets of pairs of rational numbers,

$$c(k, n) = \{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\}$$

with the following property. Let $E_{c(k,n)}$ be set of real numbers in the union of the open intervals whose endpoints are the pairs in $c(k, n)$,

$$E_{c(k,n)} = \bigcup_{(a,b) \in c(k,n)} (a, b)$$

Then,

- $E_{c(k,n)}$ is included in $E_{c(k,n-1)}$ and
- measure of $E_{c(k,n)}$ is $1 - \frac{1}{k} + \frac{1}{k+n}$

Finally, for each k,

$$E(k) = \bigcap_n E_{c(k,n)} \text{ has measure exactly } 1 - 1/k$$

and it consists entirely of normal numbers.

Main idea in Turing's Theorem 1: finite approximations

The construction is uniform in the parameter k.

Fix k. Prune the unit interval, by stages.

Stage 0: $E_{c(k,0)}$ is the whole unit interval.

Stage n: $E_{c(k,n)}$ results from removing from $E_{c(k,n-1)}$ the points that are **not** candidates to be normal, according to the inspection of an **initial segment** of their expansions.

Main idea in Turing's Theorem 1: finite approximations

At the end, the construction discards

- all rational numbers, because of their periodic expansion

Main idea in Turing's Theorem 1: finite approximations

At the end, the construction discards

- all rational numbers, because of their periodic expansion
- all irrational numbers with an unbalanced expansion

Main idea in Turing's Theorem 1: finite approximations

At the end, the construction discards

- all rational numbers, because of their periodic expansion
- all irrational numbers with an unbalanced expansion
- all normal numbers whose convergence to normality is too slow.

For each k, $E(k) = \bigcap_n E_{c(k,n)}$ consists entirely of normal numbers.
Its measure is exactly $1 - 1/k$

Main idea in Turing's Theorem 1: finite approximations

Computable functions of the stage n,

initial segment size	linear
base	sublinear
block length	sublogarithmic
frequency discrepancy ...	the technically largest converging to 0

$E_{c(k,n)}$, the set of reals not discarded up to stage n, is the union of finitely many intervals, tailored to measure $1 - \frac{1}{k} + \frac{1}{k+n}$.

Constructive Strong Law of Large Numbers

In most initial segments:

each single digit occurs about the expected number of times
each block of two digits occurs about the expected number of times
...
each block short-enough occurs about the expected number of times.

Lemma (extends Hardy & Wright 1938)

Fix b, w of length ℓ and N. For any real ε such that $\frac{7}{N} \leq \varepsilon \leq \frac{1}{b^\ell}$,

$$\sum_{\left|\frac{i}{N}-\frac{1}{b^\ell}\right|\geq\varepsilon} \begin{array}{l}\textit{number of blocks of length } N\\ \textit{with exactly } i \textit{ occurrences of } w\end{array} \leq b^N\, 2\, b^{2\ell} e^{-\frac{b^\ell \varepsilon^2 N}{6\ell}}.$$

Turing's Theorem 2

Theorem 2

There is a rule whereby given an integer K and a infinite sequence of figures 0 and 1 (the p th figure in the sequence being $\vartheta(p)$) we can find a normal number $\alpha(K,\vartheta)$ in the interval (0,1) and in such a way that for fixed K these numbers form a set of measure at least $1 - 2/K$, and so that the first n figures of ϑ determine $\alpha(K,\vartheta)$ to within 2^{-n} .

Turing's Theorem 2

There is an algorithm that, given an integer k and an infinite sequence ν of zeros and ones, produces a normal number $\alpha(k, \nu)$ in the unit interval, expressed in base two.

Turing's Theorem 2

There is an algorithm that, given an integer k and an infinite sequence ν of zeros and ones, produces a normal number $\alpha(k, \nu)$ in the unit interval, expressed in base two.

In order to write down the first n digits of $\alpha(k, \nu)$ the algorithm requires at most the first n digits of ν.

Turing's Theorem 2

There is an algorithm that, given an integer k and an infinite sequence ν of zeros and ones, produces a normal number $\alpha(k, \nu)$ in the unit interval, expressed in base two.

In order to write down the first n digits of $\alpha(k, \nu)$ the algorithm requires at most the first n digits of ν.

For a fixed k these numbers $\alpha(k, \nu)$ form a set of measure at least $1 - 2/k$.

About representation of numbers

A dyadic interval $I = \left(\frac{a}{2^n}, \frac{a+1}{2^n}\right)$, where $a \in \{0, \ldots, 2^n - 1\}$.

Hence if $x \in I$, the first n digits of x are the base 2 representation of a.

For example, for every $x \in \left(\frac{1}{2^3}, \frac{2}{2^3}\right)$ we have $x = 0,001....$

The idea in Theorem 2: "follow the measure"

Given k and ν. The algorithm works by steps.

Step 0, let $I_0 = (0, 1)$.

At step n define a dyadic interval I_n,
which is either the left half or the right half of I_{n-1}.

Output $\alpha(k, \nu) = \bigcap_{n \geq 0} I_n$.

The idea in Theorem 2: “follow the measure”

Start with the unit interval: $I_0 = (0, 1)$

The idea in Theorem 2: "follow the measure"

Start with the unit interval: $I_0 = (0, 1)$

At each step n, divide the current interval I_{n-1} in two halves, and choose I_n as the half that includes normal numbers in large-enough measure.

The idea in Theorem 2: "follow the measure"

Start with the unit interval: $I_0 = (0, 1)$

At each step n, divide the current interval I_{n-1} in two halves, and choose I_n as the half that includes normal numbers in large-enough measure.

If both halves do, use the current bit of the oracle to decide
(this will happen infinitely often)

The output $\alpha(k, \nu)$ is the trace of the left/right selection at each step.

Algorithm

4

With each integer n we associate an interval of the form $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ whose intersection with $E_{(K)}$ is of positive measure. and given m_n we obtain m_{n+1} as follows. Put

$$m\, E_{c(K,n)} \cap \left(\frac{m_n}{2^n}, \frac{2m_n+1}{2^{n+1}}\right) = a_{n,m}$$

$$m\, E_{c(K,n)} \cap \left(\frac{2m_n+1}{2^{n+1}}, \frac{m_n+1}{2^n}\right) = b_{n,m}$$

and let r_n be the smallest m for which either $a_{n,m} < K^{-1}2^{-2n}$ or $b_{n,m} < K^{-1}2^{-2n}$ or both $a_{n,m} > \frac{1}{K(K+n+1)}$ and $b_{n,m} > \frac{1}{K(K+n+1)}$. There exists such an r_n for $a_{n,m}$ and $b_{n,m}$ decrease either to 0 or to some positive number. In the case where $a_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n + 1$: if $a_{n,r_n} \geq K^{-1}2^{-2n}$ but $b_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n$, and in the third case we put $m_{n+1} = 2m_n$ or $m_{n+1} = 2m_n + 1$ according as $\vartheta(n) = 0$ or 1. For each n the interval $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^{n+1}}\right)$ includes normal numbers in positive measure. The intersection of these intervals contains only one number which must be normal.

Correctness of the algorithm

- Invariant: $I_n \cap E(k)$ has positive measure.
- Threshold: measure $(I_n \cap E_{c(k,n)})$ is larger than $M(k,n)$, where

$$2M(k,n) = M(k,n-1) - \big(\text{measure} E_{c(k,n-1)} - \text{measure} E_{c(k,n)}\big).$$

- Output: $\alpha(k,n) = \bigcap_n I_n$, with explicit convergence to normality.

Turing's normal numbers

By taking particular instances of the input sequence ν the set of numbers that can be output has measure at least $1 - 2/k$.

When ν is computable (Turing puts all zeros), the algorithm yields a computable normal number.

The algorithm can be adapted to intercalate the bits of ν at fixed positions of the output sequence.

Theorem (Figueira PhD Thesis 2006)

There is a normal number in each Turing degree.

Computational Complexity of Turing's algorithm

The number of operations to produce a next digit in the output

- *simple-exponentially* many (literal reading)
- *double-exponentially* many (our reconstruction)

Turing's First Page of the Handwritten Manuscript

Not transcribed.

His own appraisal of his work.

A Note on Normal Numbers

A. M. Turing

Although it is known that all numbers are normal 1) no example of a normal number has ever been given. I propose to show how normal numbers can be constructed and ~~[illegible]~~ for each real ε>0 ~~[illegible]~~ to give a rule by which a normal number is assigned to each real number between 0 and 1, the measure of the set of these normal numbers exceeding 1−ε.

It may be noticed that ~~[illegible]~~ ~~Champernowne~~ examples of numbers normal in the scale of 10 have already been given by Champernowne 2). It may also be noticed that ~~the~~ the normal numbers here described ~~[illegible]~~ are not [illegible]) written down ~~[illegible]~~ ~~the difficulty~~ ~~[illegible]~~ of this note cannot therefore be considered as providing convenient examples of normal numbers, but rather argues a ~~[illegible]~~ contrary ~~argument~~ to the opposition that the construction proof for normal numbers provides no examples of them. The argument is the rule for which ~~[illegible]~~ ~~[illegible]~~ [illegible] to construct proofs [illegible] closely.

Turing's First Page of the Handwritten Manuscript

"No example of a normal number has ever been given."

Turing cites Champernowne's 0.123456789101112131415...
as an example of a normal number in base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers"//

Turing's First Page of the Handwritten Manuscript

"No example of a normal number has ever been given."

Turing cites Champernowne's 0.123456789101112131415...
as an example of a normal number in base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>*convenient*</u> *examples of normal numbers"//*

He was aware of the algorithm's computational complexity.

Turing's First Page of the Handwritten Manuscript

"No example of a normal number has ever been given."

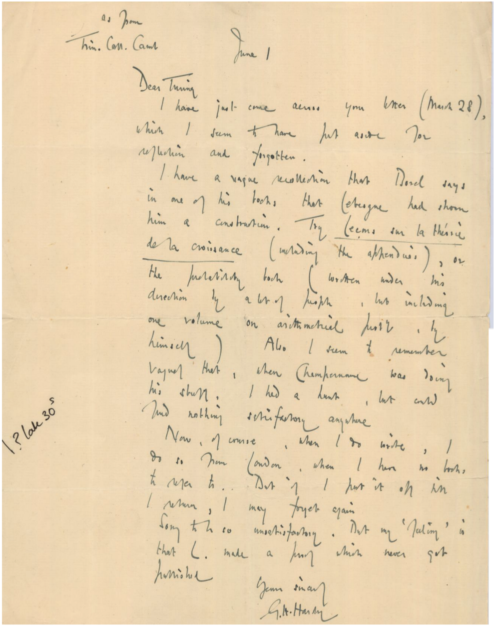
Turing cites Champernowne's 0.123456789101112131415...
as an example of a normal number in base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers"//

He was aware of the algorithm's computational complexity.

//"but rather, to counter [...] that the existence proof of normal numbers provides no example of them. The arguments in the note, in fact, follow the existence proof fairly closely."

Letter exchange between Turing and Hardy (AMT/D/5)



as from
Trin. Coll. Camb

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the probability book (written under his direction by a lot of people, but including one volume on arithmetical prob., by himself). Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere.

1? late 30s

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till I return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published.

Yours sincerely
G.H. Hardy

June 1

Dear Turing,

I have just came across your letter (March 28) which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the productivity book (written under his direction by a lot of people, but including one volume on arithmetical prosy, by himself).

Also I seem to remember vaguely that when Champernowne was doing his stuff I had a hunt, but could not find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till my return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that Lebesgue made a proof which never got published.

Yours sincerely,

G.H. Hardy

G.H. Hardy was right but he missed the novelty

Henri Lebesgue in 1909

Waclaw Sierpiński in 1916

independently, each gave a non-finitary based construction:

Bulletin de la Société Mathématique de France 45:127–132 and 132–144, 1917

Turing's Note on Normal Numbers

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

Turing's Note on Normal Numbers

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

Gave a better answer to Borel's question: an algorithm!
(unfortunately exponential)

Turing's Note on Normal Numbers

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

Gave a better answer to Borel's question: an algorithm!
(unfortunately exponential)

Started effective mathematics: concepts specified by finitely definable approximations could be made computational.

Turing's Note on Normal Numbers

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

Gave a better answer to Borel's question: an algorithm!
(unfortunately exponential)

Started effective mathematics: concepts specified by finitely definable approximations could be made computational.

In particular, Turing pioneered the theory of algorithmic randomness.

Latest results on algorithms for absolutely normal numbers

- Nearly quadratic time algorithm
  In 2013 :Figueira, Nies; Becher, Heiber and Slaman; Lutz, Mayordomo.
- 2021 Nearly linear time algorithm
  Lutz, Mayordomo, 2021.
- Discrepancy smaller than that almost all numbers
  Aistleitner, Becher, Scheerer, Slaman 2017
- Combination of normality with other properties
  Liouville, continued fraction normal, controlling different bases, a number and its reciprocal.
  Becher, Salman; Becher, Bugeaud, Slaman; Becher, Heiber, Slaman
- Descriptive complexity of the set of normal nubers, and related sets.
  Ki, Linton 1994; Becher, Heiber 2014; Slaman; Becher, Slaman 2016; the series by Airey, Jackson, Kwietniak, Mance.

References

Turing, A. M. A Note on Normal Numbers. *Collected Works of Alan M. Turing, Pure Mathematics*, edited by J. L. Britton, 117-119. Notes of editor, 263–265. North Holland, 1992.
Reprinted in *Alan Turing - his work and impact*, S B. Cooper and J. van Leeuwen editors, Elsevier, 2012.

Becher,V., Figueira,S. 2002. An example of a computable absolutely normal number. *Theoretical Computer Science* 270:947–958.

Becher,V., Figueira,S., Picchi,R. 2007. Turing's unpublished algorithm for normal numbers. *Theoretical Computer Science* 377:126–138.

C. Aistleitner, V. Becher, A.-M. Scheerer, T. Slaman,2017. On the construction of absolutely normal numbers Acta Arith., 180 (4) pp. 333-346
Becher,V., Heiber,P., Slaman,T., 2013. A polynomial-time algorithm for computing absolutely normal numbers, *Information and Computation*, 232, pp. 1-9
Borel, É. 1909. Les probabilités dénombrables et leurs applications arithmétiques. *Rendiconti del Circolo Matematico di Palermo* 27:247–271.

Champernowne, D. The Construction of Decimals in the Scale of Ten. *Journal of the London Mathematical Society*, volume 8, 254-260, 1933.

Hardy,G.H., Wright,E.M. 1938 fiirst edition. *An Introduction to the Theory of Numbers*. Oxford University Press.

Lutz, J and Mayordomo E. Computing absolutely normal numbers in nearly linear time *Information and Computation* 104746, 2021

Lebesgue, H. 1917. Sur certaines démonstrations d'existence. *Bulletin de la Société Mathématique de France* 45:132–144.

Martin-Löf, P. 1966. The Definition of Random Sequences. *Information and Control* 9(6): 602–619.

Sierpiński, W. 1917. Démonstration élémentaire du théorème de M. Borel sur les nombres absolument normaux et détermination effective d'un tel nombre. *Bulletin de la Société Mathématique de France* 45:127–132.

Turing,A.M. 1936. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society* Series 2, 42:230–265.