# Turing's Normal Numbers

Verónica Becher

Universidad de Buenos Aires

CMU math logic seminar, November 29, 2022

A Note on Nucle Nules When this figure has been calculated and written do .8 that rigure of a they treedtion to Arched. Hene comple of a want autor has even him given git for w let K be the D.N of A . What does A do in the K t hot on he It atte test whether K is satisfactory giving het U and I he verdict cannot be & hand the other hand the verdi For if it were, then in the kith section of its motion 3-1 prove the first  $\mathcal{R}(k-1) + l = \mathcal{R}(k)$  figures of the computed by the machine with / as its D.N and to write

Verónica Becher



Verónica Becher

**Turing's Normal Numbers** 

#### A Note on Normal Numbers

Although it is known that almost all numbers are normal <sup>1)</sup> no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the  $\mathcal{R}$  -figure integers in the scale of  $\mathcal{L}(t,7,2)$ . If  $\gamma$  is any sequence of figures in that scale we denote by  $N(t,\gamma)$ , the number of these in which  $\gamma$  occurs exactly m times. Then it can be proved without difficulty that

 $\frac{\sum_{k=1}^{R} N(t, \gamma, n, R)}{\sum_{k=1}^{R} N(t, \gamma, n, R)} = \frac{R - r + 1}{R} t - r$ where  $l(\gamma) = r$  is the lenght of the sequence  $\gamma$ : it is also
Veronica Becher Turing's Normal Numbers 3/34

#### A Note on Normal Numbers.

Although it is known that almost all numbers are normal <sup>21</sup> no example of a normal number has ever been given , I propose to show hav normal numbers may be occurrated and to prove that almost all numbers are schemel constructively

Consider the  ${\mathcal R}$  -figure integers in the scale of F ( F,Z,L). If  $\gamma$  is any requestor of figures in that scale we denote by  $N(\ell_{1},\mu_{1},m,R)$  the number of theorem which  $\gamma$  occurs exactly so times. Then it can be parent without difficulty that

$$\frac{\sum_{n \in \mathbb{Z}} h M(t_i | \mathbf{y}, \mathbf{n}, \mathbf{R})}{\frac{\mathbf{R}}{2} - N(t_i | \mathbf{y}, \mathbf{n}, \mathbf{R})} \approx \frac{\mathbf{R} - \mathbf{r} + \mathbf{I}}{\mathbf{R}} t^{-\mathbf{r}}$$

where  $d_{\gamma}^{n+1}$ : r is the length of the sequence  $\gamma$  : it is also possible<sup>4</sup> to prove that

$$\sum_{|n-\overline{n}t^{-r}|>R} N(t,y,n,R) (2t^{-R}e^{-N^{2}/4R} + K^{2}(2))$$

Let  $\alpha$  be a real number and  $S(\alpha, t, \gamma, R)$  the number of cocurrences of  $\gamma$  in the first R figures after the desiral point in the surrowation of  $\alpha$  in the scale of t.  $\alpha$  is call to be normal if

store r. Gyl .

Now consider sums of a fluite number of open intervals with rational cat points. These can be exampled constructively. We take a particular constructive examples (a) for  $k_{a}$  be the  $\omega$  th

 $\underline{\gamma}_{\rm R}=\underline{\gamma}_{\rm R}+\underline{1}$  whose intersection with  $\underline{\gamma}_{\rm R}^{\rm c}$  is of positive measure .

$$\begin{array}{ccc} k_{\mathbf{f}} & \tilde{h}_{\mathbf{f}}^{*}(\boldsymbol{f}_{1}(\boldsymbol{a}) + \left(\frac{m_{\mathbf{f}}}{2^{\mathbf{h}}}, \frac{2m_{\mathbf{f}}^{*}}{2^{\mathbf{h}^{*}\mathbf{f}}}\right) \leq & \tilde{h}_{\mathbf{h}_{1},\mathbf{m}}, \\ & & \tilde{h}_{\mathbf{h}_{1}}^{*}(\boldsymbol{a}) + \left(\frac{2m_{\mathbf{h}_{1}}+1}{2^{\mathbf{h}^{*}\mathbf{f}}}, \frac{m_{\mathbf{h}_{2}}+1}{2^{\mathbf{h}^{*}\mathbf{f}}}\right) \leq & \tilde{h}_{\mathbf{h}_{2},\mathbf{m}}, \end{array}$$

$$\begin{split} &h^{-1} \left( \mathcal{L}_{i_{i_{j}}}^{-1} \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \left( \frac{1}{1-1} \right) - \frac{1}{1-1} \left( \frac{1}{1-1} \right) \left( \frac{1$$

#### which must be normal.

Now consider the set  $\widetilde{n}_i(K_{+})$  consisting of all possible intervals  $\frac{\alpha_0}{2^{n-1}}, \frac{\alpha_0+1}{2^n}$  ) i.e. the sum of all these intervals as we allow the first  $\kappa$  figures of  $\mathcal{A}^0$  to run through all possibilities. Then

 $\underset{(H_{i}) \land A}{=} \frac{\mathcal{H}_{i}(H_{i} \land n + 2)}{\mathcal{H}_{i}(H_{i} \land n + 2)} := \kappa \frac{\mathcal{L}_{(H_{i}) \land A}}{\mathcal{L}_{i}(H_{i} \land n)} \frac{\mathcal{H}_{i}(H_{i} \land n)}{\mathcal{H}_{i}(H_{i} \land n)} - \frac{\mathcal{H}_{i}(H_{i} \land n)}{\mathcal{H}_{i}(H_{i} \land$ 

est of intervals in the anuseration. Then we have Theorem 1 We can find a constructive<sup>3</sup> function c(X, u) of two integral variables, such that and a E (Max) > 1 - 1/ for each H , a and  $\hat{E}_{(H)} : \prod_{i=1}^{H} \hat{E}_{i(H,u)}$  consists entirely of normal support for ench K . Let  $\mathbb{B}(d, \gamma, \ell, \mathcal{R})$  be the set of numbers  $\mathbf{x}_{i}$  (for which (5/4, t, y, R) - R++ K AFT then by (1) we B  $(\Delta, k, k, R) > 2 - 4e^{-R_{AD}^{k}}$ 2 Marit Let  $H(\Delta, T, L, R)$  be the set of three X for which (2) holds HOMMER 2556T and lig) 6 L 1.0.  $\mathcal{H}(\mathcal{A}, \tilde{\tau}, L, \mathcal{R}) := \frac{T}{11} \frac{1}{11} \frac{1}{11} \mathcal{B}(\mathcal{A}, \chi, \ell, \mathcal{R})$ The number of fectors is the product is at most  $T^{L+2}$  so that ~ A(A,T,L,R) > 1 - T LOI OF OUTO - ST H. = A([k1+], [c Jigt], [Jugk - ]], k) A. A/k I [etter, [tak-2], ke)



then if  $k\gtrsim prop}$  we shall have  $\approx \theta_k > 2 - k\,e^{-\frac{1}{2}\,k^{N_k}} > 1 - \frac{\ell}{k(k-2)}$   $\mathcal{L}(H_1\mu)$  (  $H\gtrsim rev0$ ) is to be defined as follows

#### C(K,O) 10 (0,1)

#### Theorem 2

There is a rule whereby gives an integer  $\eta$  infinite operators of 11 the  $\eta$  th figure to the sequence band 11 the  $\eta$  th figure in the sequence band  $\eta(\eta,\eta)$  are solved number  $\eta(\eta,\eta)$  but has been the sequence band,  $\eta(\eta,\eta)$  and is such as any that for final  $\eta$  three support time s are of measure at least  $1 = \frac{R}{2}/R$ , and so that for since  $\pi$ . (A) for some the state  $\pi$  is state  $\pi$ .

Verónica Becher

**Turing's Normal Numbers** 

## A Note on Normal Numbers, Alan M. Turing

Written presumably in 1936.

Unpublished until 1992, when included in the Collected Works edited by J.L.Britton. An editorial note says that the proof of the second theorem is inadequate and speculates that the theorem could be false.

## A Note on Normal Numbers, Alan M. Turing

Written presumably in 1936.

Unpublished until 1992, when included in the Collected Works edited by J.L.Britton. An editorial note says that the proof of the second theorem is inadequate and speculates that the theorem could be false.

Reconstructed, corrected and completed in 2007 Becher, Figueira, Picchi, *Theoretical Computer Science* 377, 126-138.

## Normality, a form of randomness

Defined by Émile Borel in 1909, 1922:

### Normality, a form of randomness

Defined by Émile Borel in 1909, 1922:

A real number is *normal to a given integer base* if in its expansion in that base every block of digits of the same length occurs with the same limit frequency.

For instance, if a number is normal to base 2, each of the digits '0' and '1' occur in the limit, half of the times; each of the blocks '00', '01', '10' and '11' occur one fourth of the times, and so on.

## Normality, a form of randomness

Defined by Émile Borel in 1909, 1922:

A real number is *normal to a given integer base* if in its expansion in that base every block of digits of the same length occurs with the same limit frequency.

A real number that is normal to every integer base is called *absolutely normal*, or just *normal*.

#### Counterexamples

#### 0.10100100010000100001... not normal to base 2.

#### $0.10100100010000100000\ldots$ not normal to base 2.

0.10101010101010101010101... not normal to base 2.

0.10100100010000100001... not normal to base 2.

0.10101010101010101010101... not normal to base 2.

Rationals are not normal (they have a periodic expansion in each base ).



#### Theorem (Borel 1909)

The set of normal numbers in the unit interval has Lebesgue measure 1.



#### Theorem (Borel 1909)

The set of normal numbers in the unit interval has Lebesgue measure 1.

Borel asked for an explicit example.

#### A Note on Normal Numbers

Although it is known that almost all numbers are normal <sup>1)</sup> no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the  $\mathcal{R}$  -figure integers in the scale of  $\mathcal{L}(\mathcal{L},\mathcal{I},\mathcal{I})$ . If  $\gamma$  is any sequence of figures in that scale we denote by  $\mathcal{N}(\mathcal{L},\gamma,\mathfrak{n},\mathcal{R})$  the number of these in which  $\gamma$  occurs exactly  $\mathfrak{m}$  times. Then it can be proved without difficulty that

 $\frac{\sum_{\substack{h=2\\ h=2}}^{N} N(t, \gamma, h, R)}{\sum_{\substack{h=1\\ h=1}}^{R} N(t, \gamma, h, R)} = \frac{R - r + 1}{R} t^{-r}$ where  $\ell(\gamma) = r$  is the lenght of the sequence  $\gamma$  : it is also possible<sup>2</sup>) to prove that

" RI INLR

**Turing's Normal Numbers** 

9/34

#### Turing's Theorem 1

Borel's theorem on the measure of normal numbers, constructively.

#### Turing's Theorem 2

An algorithm to construct normal numbers.

#### Turing's First Page of the Handwritten Manuscript

His own appraisal of his work.

#### Theorem 1

We can find a constructive<sup>3)</sup> function  $c(\mathcal{H}, u)$  of two integral variables, such that

$$\frac{F_{c}(K, n+1)}{m} \leq F_{c}(K, n)$$
and
$$m F_{c}(K, n) > 1 - \frac{L}{K} \quad \text{for each } K, h$$
and
$$\frac{m}{F_{c}(K)} = \frac{m}{11} F_{c}(K, n) \quad \text{consists entirely of normal numbers for}$$
each  $K$ .

There is a computable function c(k, n) of two integer numbers with values in finite sets of pairs of rational numbers,

$$c(k, n) = \{(a_1, b_1), (a_2, b_2), \dots, (a_m, b_m)\}$$

with the following property. Let  $E_{c(k,n)}$  be set of real numbers in the union of the open intervals whose endpoints are the pairs in c(k, n),

$$E_{c(k,n)} = \bigcup_{(a,b)\in c(k,n)} (a,b)$$

Then,

There is a computable function c(k, n) of two integer numbers with values in finite sets of pairs of rational numbers,

$$c(k,n) = \{(a_1, b_1), (a_2, b_2), \dots, (a_m, b_m)\}$$

with the following property. Let  $E_{c(k,n)}$  be set of real numbers in the union of the open intervals whose endpoints are the pairs in c(k, n),

$$E_{c(k,n)} = \bigcup_{(a,b)\in c(k,n)} (a,b)$$

Then,

• 
$$E_{c(k,n)}$$
 is included in  $E_{c(k,n-1)}$  and

• measure of 
$$E_{c(k,n)}$$
 is  $1 - \frac{1}{k} + \frac{1}{k+n}$ 

Finally, for each k,

$$E(k) = \bigcap_{n} E_{c(k,n)}$$
 has measure exactly  $1 - 1/k$ 

and it consists entirely of normal numbers.

The construction is uniform in the parameter k.

Fix k. Prune the unit interval, by stages.

Stage 0:  $E_{c(k,0)}$  is the whole unit interval.

Stage *n*:  $E_{c(k,n)}$  results from removing from  $E_{c(k,n-1)}$  the points that are **not** candidates to be normal, according to the inspection of an **initial segment** of their expansions.

At the end, the construction discards

> all rational numbers, because of their periodic expansion

At the end, the construction discards

- > all rational numbers, because of their periodic expansion
- all irrational numbers with an unbalanced expansion

At the end, the construction discards

- > all rational numbers, because of their periodic expansion
- all irrational numbers with an unbalanced expansion
- all normal numbers whose convergence to normality is too slow.

For each k,  $E(k) = \bigcap_{n} E_{c(k,n)}$  consists entirely of normal numbers. Its measure is exactly 1 - 1/k

Computable functions of the stage n,

initial segment size	linear
base	sublinear
block length	sublogarithmic
frequency discrepancy	the technically largest converging to $\boldsymbol{0}$

 $E_{c(k,n)}$ , the set of reals not discarded up to stage *n*, is the union of finitely many intervals, tailored to measure  $1 - \frac{1}{k} + \frac{1}{k+n}$ .

## Constructive Strong Law of Large Numbers

In most initial segments:

each single digit occurs about the expected number of times each block of two digits occurs about the expected number of times ... each block short-enough occurs about the expected number of times.

#### Lemma (extends Hardy & Wright 1938)

Fix b, w of length  $\ell$  and N. For any real  $\varepsilon$  such that  $\frac{7}{N} \le \varepsilon \le \frac{1}{b^{\ell}}$ ,

Theorem 2 Theorem 2 There is a rule whereby given an integer  $\mathcal{K}$  and  $\alpha_{\mathcal{K}}$  sequence of figures 0 and 1 ( the  $\mathcal{P}$  th figure in the sequence being  $\mathcal{N}(\mathcal{P})$  ) we can find a normal number  $\mathcal{N}(\mathcal{H}, \mathcal{P})$  in the interval (0,1) and in such a way that for fixed  $\mathcal{H}$  these numbers form a set of measure at least  $\mathcal{I} = \frac{2}{\mathcal{K}}$ , and so that the first  $\mathcal{N}$  figures of  $\mathcal{N}$  determine  $\alpha(\mathcal{K}, \mathcal{P})$ to within  $\mathcal{Q}^{-\mathcal{N}}$ . There is an algorithm that, given an integer k and an infinite sequence  $\nu$  of zeros and ones, produces a normal number  $\alpha(k,\nu)$  in the unit interval, expressed in base two.

There is an algorithm that, given an integer k and an infinite sequence  $\nu$  of zeros and ones, produces a normal number  $\alpha(k,\nu)$  in the unit interval, expressed in base two.

In order to write down the first *n* digits of  $\alpha(k, \nu)$  the algorithm requires at most the first *n* digits of  $\nu$ .

There is an algorithm that, given an integer k and an infinite sequence  $\nu$  of zeros and ones, produces a normal number  $\alpha(k,\nu)$  in the unit interval, expressed in base two.

In order to write down the first *n* digits of  $\alpha(k, \nu)$  the algorithm requires at most the first *n* digits of  $\nu$ .

For a fixed k these numbers  $\alpha(k,\nu)$  form a set of measure at least 1-2/k.

## About representation of numbers

A dyadic interval 
$$I = \left(\frac{a}{2^n}, \frac{a+1}{2^n}\right)$$
, where  $a \in \{0, \dots, 2^n - 1\}$ .  
Hence if  $x \in I$ , the first *n* digits of *x* are the base 2 representation of *a*.  
For example, for every  $x \in \left(\frac{1}{2^3}, \frac{2}{2^3}\right)$  we have  $x = 0, 001....$ 

Given k and  $\nu$ . The algorithm works by steps.

Step 0, let  $I_0 = (0, 1)$ .

At step *n* define a dyadic interval  $I_n$ , which is either the left half or the right half of  $I_{n-1}$ .

Output  $\alpha(k,\nu) = \bigcap_{n\geq 0} I_n$ .

## The idea in Theorem 2: "follow the measure"

Start with the unit interval:  $I_0 = (0, 1)$ 

Start with the unit interval:  $I_0 = (0, 1)$ 

At each step n, divide the current interval  $I_{n-1}$  in two halves, and choose  $I_n$  as the half that includes normal numbers in large-enough measure.

Start with the unit interval:  $I_0 = (0, 1)$ 

At each step n, divide the current interval  $I_{n-1}$  in two halves, and choose  $I_n$  as the half that includes normal numbers in large-enough measure.

If both halves do, use the current bit of the oracle to decide (this will happen infinitely often)

The output  $\alpha(k, \nu)$  is the trace of the left/right selection at each step.

#### Algorithm

With each integer h we associate an interval of the form  $\begin{pmatrix} \frac{m_a}{2^{m_a}}, \frac{m_a+1}{2^{m_a}} \\ \text{and given} \end{pmatrix}_{m_a} \text{ we obtain } \frac{m_a}{4+1} \text{ as follows. Put}$  $m \quad F_{c(H,u)} \land \left(\frac{m_{u}}{2^{u}}, \frac{2m_{u}+2}{2^{u+2}}\right) \simeq a_{u,u}$  $m \stackrel{\text{E}}{=} \left( K, \mu \right) \cap \left( \frac{2m_{\mu} + 7}{2^{\mu+2}}, \frac{m_{\mu} + 7}{7^{\mu}} \right) = b_{\mu_{\mu}} m$ and let  $V_{\mu}$  be the smallest m for which either  $a_{\mu,m} \leq K^{-1} 2^{-2\mu}$ or  $b_{\mu,m} \leq K^{-1} 2^{-d\mu}$  or both  $a_{\mu,m} \geq \frac{2}{\mathcal{H}(\mathcal{K}+\mu+1)}$  and  $b_{\mu,m} \geq \frac{2}{\mathcal{H}(\mathcal{K}+\mu+2)}$ There exists such an  $V_{\mu}$  for  $a_{\mu,m}$  and  $b_{\lambda,m}$  decrease either to 0 or to some positive number. In the case where  $a_{n,k} \leq K^2 2^{-d_n}$  we put  $m_{n+1} = 2m_n + 1$  : if  $a_{n,k} \geq K^2 2^{-2n}$  but  $b_{n,k} \leq K^2 2^{-d_n}$ we put  $M_{n+1} = 2m_h$ , and in the third case we put  $M_{n+1} = 2m_h$ or  $M_{\mu+1} = 2M_{\mu} + 2$  according as  $\mathcal{N}(\mu) = 0$  or 1. For each  $\mu$  the interval  $\left(\frac{m_{h}}{2h}, \frac{m_{h}+2}{2h}\right)$  includes normal numbers in positive measure. The intersection of these intervals contains only one numberwhich must be normal.

#### Correctness of the algorithm

- Invariant:  $I_n \cap E(k)$  has positive measure.
- ▶ Threshold: measure  $(I_n \cap E_{c(k,n)})$  is larger than M(k, n), where

$$2M(k,n) = M(k,n-1) - (\text{measure}E_{c(k,n-1)} - \text{measure}E_{c(k,n)}).$$

• Output: 
$$\alpha(k, n) = \bigcap_{n} I_{n}$$
, with explicit convergence to normality.

By taking particular instances of the input sequence  $\nu$  the set of numbers that can be output has measure at least 1-2/k.

When  $\nu$  is computable (Turing puts all zeros), the algorithm yields a computable normal number.

The algorithm can be adapted to intercalate the bits of  $\nu$  at fixed positions of the output sequence.

Theorem (Figueira PhD Thesis 2006)

There is a normal number in each Turing degree.

# Computational Complexity of Turing's algorithm

The number of operations to produce a next digit in the output

- simple-exponentially many (literal reading)
- double-exponentially many (our reconstruction)

Not transcribed.

His own appraisal of his work.

A Note on Wand Makes of steps, When this fight has been calculated and written down as the ender the server of the state in the all makes in the shared it is no veryle of a want actual to even him given of program w let K be the D.N of A. What does A do in the K th section a rive to the on he It what toot whether is and a single of giving a ver-- of a share and the second and the 1 the the word of all and the and have have the word of cannot the be S'. For St is vero, then in the SA action of the motion of the machine with N as its D.N and to frite down the - un clayer and wayter of work the march the such of to have tod , tolaching the begins to be Chapter and May all my about When the instructions for calculating the  $\mathcal{R}(\mathcal{X})$  , would amount to "calculate hatment the start he wanter - mos according ) . in the down of a fact it in an one but at trary both to what we have sound in the last paragraph and to the verdict and the caust have be included veriet wearply can bling We can show further that ousing the the earting a conte allo ever prints a given symbol (0 say). & and friendly who stander Windre 1s a good hads at a format a time a won sociate Commente o intinition often. Let the a machine which prints the same sequence as elle, except that in the posifion where the first our facts aft stands, ENG, , prints O. ENR, is to have the first two symbols O replaced by 0, and so on. Thus if CM were to print

Verónica Becher

"No example of a normal number has ever been given."

Turing cites Champernowne's 0.123456789101112131415... as an example of a normal number in base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers"//

"No example of a normal number has ever been given."

Turing cites Champernowne's 0.123456789101112131415... as an example of a normal number in base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers"//

He was aware of the algorithm's computational complexity.

"No example of a normal number has ever been given."

Turing cites Champernowne's 0.123456789101112131415... as an example of a normal number in base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers"//

He was aware of the algorithm's computational complexity.

// "but rather, to counter [...] that the existence proof of normal numbers provides no example of them. The arguments in the note, in fact, follow the existence proof fairly closely."

#### Letter exchange between Turing and Hardy (AMT/D/5)

as for Thin. Com. Came June 1 Dear Tring I have just me across you been (Mark 28) which I seem to have fut assore you replaching and forgotten. I have a vague recollection that Board says in me of his books that Cohegue had shown him a construction. Try learns son la théric de la croissance (whiting the appendix), or the purching both ( bothen ander this direction by abr of high , his including one volume on arithmetrich first it. himself ) Aleo I seem it remember Vayney that , when Chempername was Ising his shap. I had a hant , but and 2 lok 30 Jud rothing siniferrory anything Now, of course, when I to write, Is in non low of them I have no book, the rape to . Dor'g I have it of him I restore, I may forget equit long to the commentification. But my "Juling" in that L. make a fung shink never got honished Gem snar G.H. Hardy

Verónica Becher

Turing's Normal Numbers

June 1

Dear Turing,

I have just came across your letter (March 28) which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the productivity book (written under his direction by a lot of people, but including one volume on arithmetical prosy, by himself).

Also I seem to remember vaguely that when Champernowne was doing his stuff I had a hunt, but could not find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till my return, I may forget again. Sorry to be so unsatisfactory. But my 'feeling' is that Lebesgue made a proof which never got published. Yours sincerely,

G.H. Hardy

G.H. Hardy was right but he missed the novelty

Henri Lebesgue in 1909

Waclaw Sierpiński in 1916

independently, each gave a non-finitary based construction:

Bulletin de la Société Mathématique de France 45:127-132 and 132-144, 1917

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

Gave a better answer to Borel's question: an algorithm! (unfortunately exponential)

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

Gave a better answer to Borel's question: an algorithm! (unfortunately exponential)

Started effective mathematics: concepts specified by finitely definable approximations could be made computational.

A story of unrecognized scientific priority

Proved the existence of computable normal numbers.

Gave a better answer to Borel's question: an algorithm! (unfortunately exponential)

Started effective mathematics: concepts specified by finitely definable approximations could be made computational.

In particular, Turing pioneered the theory of algorithmic randomness.

#### Latest results on algorithms for absolutely normal numbers

- Nearly quadratic time algorithm In 2013 :Figueira, Nies; Becher, Heiber and Slaman; Lutz, Mayordomo.
- 2021 Nearly linear time algorithm Lutz, Mayordomo, 2021.
- Discrepancy smaller than that almost all numbers Aistleitner, Becher, Scheerer, Slaman 2017
- Combination of normality with other properties
   Liouville, continued fraction normal, controlling different bases, a number and its reciprocal.
   Becher, Salman; Becher, Bugeaud, Slaman; Becher, Heiber, Slaman
- Descriptive complexity of the set of normal nubers, and related sets. Ki, Linton 1994; Becher, Heiber 2014; Slaman; Becher, Slaman 2016; the series by Airey, Jackson, Kwietniak, Mance.

#### References

Turing, A. M. A Note on Normal Numbers. Collected Works of Alan M. Turing, Pure Mathematics, edited by J. L. Britton, 117-119. Notes of editor, 263–265. North Holland, 1992. Reprinted in Alan Turing - his work and impact, S B. Cooper and J. van Leeuwen editors, Elsevier, 2012.

Becher, V., Figueira, S. 2002. An example of a computable absolutely normal number. *Theoretical Computer Science* 270:947–958.

Becher, V., Figueira, S., Picchi, R. 2007. Turing's unpublished algorithm for normal numbers. *Theoretical Computer Science* 377:126–138.

C. Aistleitner, V. Becher, A.-M. Scheerer, T. Slaman,2017. On the construction of absolutely normal numbers Acta Arith., 180 (4) pp. 333-346 Becher, V., Heiber, P., Slaman, T., 2013. A polynomial-time algorithm for computing absolutely normal numbers, Information and Computation, 232, pp. 1-9 Borel, É. 1909. Les probabilités dénombrables et leurs applications arithmétiques. Rendiconti del Circolo Matematico di Palermo 27:247–271.

Champernowne, D. The Construction of Decimals in the Scale of Ten. Journal of the London Mathematical Society, volume 8, 254-260, 1933.

Hardy, G.H., Wright, E.M. 1938 fiirst edition. An Introduction to the Theory of Numbers. Oxford University Press.

Lutz, J and Mayordomo E. Computing absolutely normal numbers in nearly linear time Information and Computation 104746, 2021

Lebesgue, H. 1917. Sur certaines démonstrations d'existence. Bulletin de la Société Mathématique de France 45:132–144.

Martin-Löf, P. 1966. The Definition of Random Sequences. Information and Control 9(6): 602-619.

Sierpiński, W. 1917. Démonstration élémentaire du théorème de M. Borel sur les nombres absolument normaux et détermination effective d'un tel nombre. Bulletin de la Société Mathématique de France 45:127–132.

Turing, A.M. 1936. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society* Series 2, 42:230–265.