Alan Turing's algorithm for producing absolutely normal numbers

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A Note on Normal Numbers

A. M. Turing

Although it is known that all numbers are normal 1) no example of a normal number has ever been given. I propose to show how normal numbers can be ~~constructed~~ and ~~[illegible]~~ ~~by [illegible]~~ to give a rule by which a to each $\varepsilon > 0$ normal number is assigned to each real number between 0 and 1, the measure of the set of these normal numbers exceeding $1-\varepsilon$.

It may be noticed that ~~[illegible]~~ ~~Champernowne~~ examples of numbers normal in the scale of 10 have already been given by Champernowne 2). It may also be noticed that ~~[illegible]~~ the ~~examples of~~ normal numbers ~~given~~ described have are not real) written down ~~[illegible]~~.

~~The difficulty~~ of this note cannot therefore be considered

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Although it is known that almost all numbers are normal 1) no example of a normal number has ever been given. I propose to show how normal numbers can be constructed and ~~[illegible]~~ to prove that almost all numbers ~~[illegible]~~ between 0 and 1 are normal, ~~[illegible]~~ to give a rule by which a normal number is assigned to each real number between 0 and 1, the measure of the set of these normal numbers exceeding 1 − ε.

It may be noticed that ~~[illegible]~~ examples of numbers normal in the scale of 10 have already been given by Champernowne 2). It may also be noticed that ~~[illegible]~~ the ~~examples of~~ normal numbers here described [illegible] cannot therefore be considered ~~[illegible]~~ as providing convincing examples of normal numbers, but rather as a [illegible] to the [illegible] that the existence proof for normal numbers provides no examples of them. The [illegible] what follows [illegible] fairly [illegible] the [illegible] [illegible] [illegible].

If γ is a sequence of digits in the scale of t let
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Definition If γ is a sequence of digits in the scale of t [illegible] $l(\gamma)$ is the length of γ.

If α is a real number between 0 and 1 and γ is a sequence of digits in the scale of t we put

$S(\alpha, t, \gamma, R)$ = number of occurrences of γ in the first R figures of the decimal expansion of α in the scale of t.

α is said to be normal in the scale of t with respect to γ if

$$R^{-1} S(\alpha, t, \gamma, R) \to t^{-l(\gamma)} \text{ as } R \to \infty \qquad (1)$$

It is said to be normal in the scale of t if (1) holds for all γ, and to be normal if it holds for all γ, t.

We shall also put

$N(t, \gamma, n, R)$ = number of sequences of R digits in the scale of t in which γ occurs exactly n times.

Instead of saying α is normal with respect to γ we may say that γ has normal frequency [illegible].

Lemma 1

If a number is normal in the scale of t, with [illegible] ~~[illegible]~~ sequences which are not of the form $\beta_1 \beta_2 \beta_1$ then it is normal in the scale of t.

If not normal let us consider one of the shortest sequences β_1 in a case $\beta_1 \beta_2 \beta_1$ of failure. Let β_2 be [illegible] possible [illegible] with β_1 having the same length and [illegible] let the number of repetitions of β_2 [illegible] and of β_1 also have the [illegible] value. The sequence is

l times (say)

$$\beta_1 \beta_3 \beta_1 \beta_1 \text{—} \beta_1$$

Now $\beta_1 \beta_3$ ~~[illegible]~~ has normal frequency ~~[illegible]~~.

~~[illegible]~~ [illegible] that [illegible] the other [illegible] and so [illegible] every sequence $\beta_1 \beta_3 \beta_2$ where β_2 is different to $\beta_1 \text{—} \beta_1$ [illegible] of the same length [illegible] $\beta_1 \beta_3 \beta$ [illegible] [illegible] [illegible] $\beta_1 \beta_3 \beta_1 \text{—} \beta_1$ [illegible] [illegible] to hyp.

Lemma 2

If γ is a sequence not of the form $\beta_1 \beta_2 \beta_1$ and of length l then the number $N(t, \gamma, n, R)$ of ~~[illegible]~~ sequences of length R in which γ occurs exactly n times ~~[illegible]~~

$$N(t, \gamma, n, R) \leq \frac{(R/l)^n}{n!} t^R$$

$N(t, \gamma, n, R)$ = No of ways of choosing positions of sequences γ × No of ways of filling [illegible]

$$\leq \frac{R(R-l)(R-2l) \ldots (R-nl+l)}{n!} t^{R-nl}$$

$$\leq \frac{(R/l)^n}{n!} t^R$$

$$\sum_{|n-R t^{-l}| > k} N(t, \gamma, n, R) \leq t^{-R} \sum_{n > R t^{-l} + k} \frac{(R/l)^n}{n!}$$

$$\leq t^{-R} \frac{(R/l)^{R/l}}{[R/l]!} \ldots$$

If γ be different first and last equal digits then [illegible] γ

$N(t, \gamma, n, R)$ = No of R-figure sequences in the scale of t in which γ occurs exactly n times.

~~[illegible]~~

We have

$N(t, \gamma, n, R)$ = No of ways of placing the ~~figures~~ sequence γ × No of ways of filling in the gaps.

~~$\frac{1}{n!} t^R$~~

$$\leq \frac{R(R-l)(R-2l)(R-nl+l)}{n!} t^{R-nl}$$

[illegible] $\leq \frac{R(R-1) \ldots (R-n)}{n!} t^{R-nl}$

$$\binom{R}{n} t^{R-nl}$$

If a sequence γ has different first and last figures we can find a sequence $\beta_1 \beta_2 \beta_3$ such that β_1, β_2, γ are $\beta_1 \beta_3 \gamma$.

A Note on Normal Numbers

Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given . I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the R -figure integers in the scale of t ($t \geqslant 2$). If γ is any sequence of figures in that scale we denote by $N(t, \gamma,$ the number of thesein which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n \, N(t, \gamma, n, R)}{\sum_{n=1}^{R} N(t, \gamma, n, R)} = \frac{R - r + 1}{R} t^{-r}$$

where $l(\gamma) = r$ is the lenght of the sequence γ : it is also possible[2] to prove that

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Consider the R-figure integers in the scale of t ($t \geq 2$). If γ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of these in which γ occurs exactly n times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\,N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\,t^{-r}$$

where $l(\gamma) = r$ is the lenght of the sequence γ : it is also possible [2] to prove that

$$\sum_{|n-Rt^{-r}|>k} N(t,\gamma,n,R) < 2t^{R} e^{-k^2 t^r/4R} \quad \text{provided } \frac{kt^r}{R} < \cdot 3 \qquad (1)$$

Let α be a real number and $S(\alpha,t,\gamma,R)$ the number of occurrences of γ in the first R figures after the decimal point in the expression of α in the scale of t. α is said to be normal if

$$R^{-1} S(\alpha,t,\gamma,R) \to t^{-r} \quad \text{as } R \to \infty \text{ for each } \gamma, t$$

where $r = l(\gamma)$.

Now consider sums of a finite number of open intervals with rational end points. These can be enumerated constructively. We take a particular constructive enumeration: let E_n be the n th

set of intervals in the enumeration. Then we have

Theorem 1

We can find a constructive[3] function $c(K,n)$ of two integral variables, such that

$$E_{c(K,n+1)} \subseteq E_{c(K,n)}$$

and $m E_{c(K,n)} > 1 - \frac{1}{K}$ for each K, n

and $E_{(K)} = \prod_{n=1}^{\infty} E_{c(K,n)}$ consists entirely of normal numbers for each K.

Let $B(\Delta,\gamma,t,R)$ be the set of numbers α, $0<\alpha<1$ for which

$$\left| S(\alpha,t,\gamma,R) - Rt^{-r} \right| < \frac{R}{\Delta t^r} \quad \left(k = \frac{R}{\Delta t^r}\right) \qquad (2)$$

then by (1)

$$m\,B(\Delta,\gamma,t,R) > 1 - 2e^{-Rt^{-r}/4\Delta^2} \qquad \Delta < \cdot 3$$

Let $A(\Delta,T,L,R)$ be the set of those α for which (2) holds whenever $2 \leq t \leq T$ and $l(\gamma) \leq L$ i.e.

$$A(\Delta,T,L,R) = \prod_{t=2}^{T} \prod_{l(\gamma)\leq L} B(\Delta,\gamma,t,R)$$

The number of factors in the product is at most T^{L+1} so that

$$m\,A(\Delta,T,L,R) > 1 - T^{L+1} e^{-\frac{RT^{-L}}{4\Delta^2}}$$

Let

$$A_k = A\left([k^{1/4}], [e^{\sqrt{\log k}}], [\sqrt{\log k} - 1], k\right)$$

$$A_k = A\left(k, [e^{\sqrt{\log k}}], [\sqrt{\log k} - 1], k^4\right)$$

then if $k \geq 1000$ we shall have $m A_k > 1 - ke^{-\frac{1}{2}k^{1/2}} > 1 - \frac{1}{k(k-1)}$.
$C(K,n)$ ($K \geq 1000$) is to be defined as follows

$C(K,0)$ is $(0,1)$

$C(K,n+1)$ is the intersection of an interval $(\beta_n,1)$, $(0 \leq \beta_n < 1)$ with A_{K+n+1} and $C(K,n)$, β_n being so chosen that the measure of $C(K,n+1)$ is $1-\frac{1}{K}+\frac{1}{K+n+1}$. This is possible since the measure of $C(K,n)$ is $1-\frac{1}{K}+\frac{1}{K+n}$ and that of A_{K+n+1} is at least $1-\frac{1}{(K+n)(K+n+1)}$: consequently the measure of $C(K,n) \cap A_{K+n+1}$ is at least $1-\frac{1}{K}+\frac{1}{K+n+1}$.
If $K < 1000$ we define $C(K,n)$ to be $C(1000,n)$. $C(K,n)$ is a finite sum of intervals for each K, n. When we remove the boundary points we obtain a set of form $E_{c(K,n)}$ of measure $1-\frac{1}{K}+\frac{1}{K+n}$ ($K \geq 1000$). The intervals of which $E_{c(K,n)}$ is composed may be found by a mechanical process and so the function $c(K,n)$ is constructive. The set $E_{(K)} = \prod_{n=1}^{\infty} E_{c(K,n)}$ consists of normal numbers, for if $\alpha \in E$ then $\alpha \in A_k$, all $k > K$ ($K \geq 1000$). If γ is a sequence of length r in the scale of t and if k_0 be such that $[e^{\sqrt{\log k_0}}] > t$ and $[\sqrt{\log k_0}] > r+1$ then for $k > k_0$

$$\left| S(\beta,t,\gamma,k) - kt^{-r} \right| < k\,[k^{1/4}]^{-1} \quad \text{when } \beta \text{ is in } A_k$$

(by the definition of A_k). Hence $k^{-1} S(\alpha,t,\gamma,k) \to t^{-r}$ as k tends to infinity, i.e. α is normal.

Theorem 2

There is a rule whereby given an integer K and an infinite sequence of figures 0 and 1 (the p th figure in the sequence being $\vartheta(p)$) we can find a normal number $\alpha(K,\vartheta)$ in the interval (0,1) and in such a way that for fixed K these numbers form a set of measure at least $1 - 2/K$, and so that the first n figures of ϑ determine $\alpha(K,\vartheta)$ to within 2^{-n}.

With each integer n we associate an interval of the form $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ whose intersection with $E_{(K)}$ is of positive measure, and given m_n we obtain m_{n+1} as follows. Put

$$m\,E_{c(K,m)} \cap \left(\frac{m_n}{2^n}, \frac{2m_n+1}{2^{n+1}}\right) = a_{n,m}$$

$$m\,E_{c(K,m)} \cap \left(\frac{2m_n+1}{2^{n+1}}, \frac{m_n+1}{2^n}\right) = b_{n,m}$$

and let r_n be the smallest m for which either $a_{n,m} < K^{-1}2^{-2n}$ or $b_{n,m} < K^{-1}2^{-2n}$ or both $a_{n,m} > \frac{1}{K(K+n+1)}$ and $b_{n,m} > \frac{1}{K(K+n+1)}$.
There exists such an r_n for $a_{n,m}$ and $b_{n,m}$ decrease either to 0 or to some positive number. In the case where $a_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n+1$: if $a_{n,r_n} \geq K^{-1}2^{-2n}$ but $b_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n$, and in the third case we put $m_{n+1} = 2m_n$ or $m_{n+1} = 2m_n+1$ according as $\vartheta(n) = 0$ or 1. For each n the interval $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^{n+1}}\right)$ includes normal numbers in positive measure. The intersection of these intervals contains only one number which must be normal.

Now consider the set $A(K,n)$ consisting of all possible intervals $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ i.e. the sum of all these intervals as we allow the first n figures of ϑ to run through all possibilities. Then

$$m\,E_{(K)} \cap A(K,n+1) = m\,E_{(K)} \cap A(K,n) - \sum_{m=0}^{2^n-1} m\,E_{(K)} \cap \left(A(K,n) - A(K,n+1)\right) \cap \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$$

For

so that

$$m\,E_{(K)} \cap A(K,n+1) > m\,E_{(K)} \cap A(K,n) - 2^{-n-1}K^{-1} > m\,E_{(K)} - K^{-1} > 1 - 2/K$$

The set of alll possible numbers $\alpha(K,\vartheta)$ is therefore of measure at least $1 - 2/K$.

By taking particular sequences ϑ (e.g. $\vartheta(n) = 0$ all n) we obtain particular normal numbers.

Turing's Note on Normal Numbers

1937 Alan M. Turing's writes the manuscript "A Note on Normal Numbers".

1992 The manuscript was included in the *Collected Works of Alan Turing*, volume *Pure Mathematics*, 1992, edited by J.L.Britton. An editorial note, page 264,
"[7] The proof of this theorem that is given is certainly inadequate. Indeed I suspect that the theorem is false. "

2007 The manuscript was reconstructed, corrected and completed
Becher, Figueira, Picchi, *Theoretical Computer Science* 377: 126-138, 2007.

For a real number x, its expansion in an integer base $b \geq 2$ is a sequence of integers $a_1, a_2 \ldots$, where $0 \leq a_n < b$ for every n, such that

$$x - \lfloor x \rfloor = \sum_{n \geq 1} a_n b^{-n} = 0.a_1 a_2 a_3 \ldots$$

We require that $a_n < b - 1$ infinitely often to ensure that every number has a unique representation.

Borel normal numbers

A real number is normal to an integer base $b \geq 2$ if in its base-b expansion all blocks of the same length occur with the same limit frequency.

$0.0100100010000100000100000...$ not normal to base 2.

A real number that is normal to every integer base is called absolutely normal, or just normal.

In 1909, Borel defined a number as simply normal to base b if in its base b-expansion every digit in $\{0, \ldots b-1\}$ occurs with equal limit frequency. A number is normal to base b if is is normal to all the bases b^k, for $k \geq 1$. In 1922, Borel provided an alternative formulation of normality in terms of the equifrequency of blocks of digits. The equivalence of these definitions was proved by Niven and Zuckerman in 1951.

Borel (1909) proved that the set of normal numbers in the unit interval has Lebesgue measure 1, and he asked for an explicit example.

Turing's note on normal numbers

Turing's Theorem 1

Borel's theorem on the Lebesgue measure of normal numbers, constructively.

Turing's Theorem 2

An algorithm to construct normal numbers.

Turing's Theorem 1

Theorem 1

We can find a constructive[3] function $c(K,n)$ of two integral variables, such that

$$E_{c(K,n+1)} \subseteq E_{c(K,n)}$$

and $\quad m\, E_{c(K,n)} > 1 - \frac{1}{K} \quad$ for each K, n

and $\quad E_{(K)} = \prod_{n=1}^{\infty} E_{c(K,n)} \quad$ consists entirely of normal numbers for each K.

Notation: $E_{\{(a_1,b_1),\ldots(a_m,b_m)\}} = \bigcup_{i=1}^{m}(a_i, b_i)$, and μ is Lebesgue measure.

Theorem (Turing's Theorem 1)

There is a computable function $c(k, n)$ of two integer variables with values in finite sets of pairs of rational numbers such that for each k and n

$$E_{c(k,n+1)} \subseteq E_{c(k,n)}, \qquad \mu E_{c(k,n)} > 1 - \frac{1}{k},$$

and

$$E(k) = \bigcap_n E_{c(k,n)}, \qquad \mu(E(k)) = 1 - \frac{1}{k}.$$

consists entirely of normal numbers.

The class of computable functions is the smallest class of functions $\mathbb{N} \to \mathbb{N}$ that contains the constant, the projections,the successor, and closed by composition, recursion and the unbounded minimization. Equivalently, it is the set of functions carried by a Turing machine.

Proof of Turing's Theorem 1

The construction is uniform in the parameter k.

The construction prunes the unit interval, by steps.

step 0: $E_{c(k,0)}$ is the whole unit interval.

step n: $E_{c(k,n)}$ results from removing from $E_{c(k,n-1)}$ the points that are not candidates to be normal, according to the inspection of an initial segment of their expansions.

At the end, the construction discards

- all rational numbers, because of their periodic structure.
- all irrational numbers with an unbalanced expansion.
- all normal numbers whose convergence to normality is too slow.

There are a few bad numbers

Let $b \geq 2$. In most sequences of sufficiently large length P every $w \in \{0, \ldots b-1\}^\ell$ occurs approximately $Pb^{-\ell}$ times.

Lemma (extends Hardy & Wright 1938)

Let $b \geq 2$, $P \geq 1$, $\ell \geq 1$, $w \in \{1, \ldots b-1\}^\ell$ and ε such that $\dfrac{7}{P} \leq \varepsilon \leq \dfrac{1}{b^\ell}$,

$$\left(\sum_{|i - Pb^{-\ell}| \geq \varepsilon P} \begin{array}{l} \textit{number of length-}P \textit{ sequences} \\ \textit{with exactly } i \textit{ occurrences of } w \end{array} \right) \leq b^P \; 2 \; b^{2\ell} e^{-\frac{b^\ell \varepsilon^2 P}{6\ell}}.$$

x_b is the expansion of x in base b, and $x_b \upharpoonright n$ is the first n digits of x_b.
$|y|_w$ is the number of occurrences of block w in the finite sequence y.

A number x is normal to base b if $\forall \ell \geq 1 \ \forall w \in \{0, \ldots b-1\}^\ell$,

$$\lim_{P \to \infty} \frac{|x_b \upharpoonright P|_w}{P} = b^{-\ell}$$

That is, $\forall \varepsilon > 0 \ \forall \ell \geq 1 \ \forall w \in \{0, \ldots b-1\}^\ell \ \exists P_0 \ \forall P \geq P_0$

$$\left| \frac{x_b \upharpoonright P|_w}{P} - \frac{1}{b^\ell} \right| < \varepsilon$$

we can rewrite as

$$\left| |x_b \upharpoonright P|_w - \frac{P}{b^\ell} \right| < P\epsilon.$$

Thus, x is normal to base b if $\forall \varepsilon > 0 \ \forall \ell \geq 1 \ \forall w \{0, \ldots b-1\}^\ell \ \exists P_0 \ \forall P \geq P_0$

$$x \notin Bad(\varepsilon, w, b, P) = \left\{ x \in (0,1) : \left| |x_b \upharpoonright P|_w - \frac{P}{b^\ell} \right| \geq \varepsilon P \right\}$$

Notice $Bad(\varepsilon, w, b, P)$ is a finite union of intervals with rational endpoints.

Define:

$$A(P) = \bigcap_{2 \le b \le B(P)} \ \bigcap_{1 \le \ell \le L(P)} \ \bigcap_{w \in \{0,\ldots,b-1\}^{L(P)}} \Big((0,1) \backslash Bad(\varepsilon(P), w, b, P) \Big)$$

where

$$\begin{array}{lll} B(P) & = e^{\sqrt{\ln P}/4} & \text{(sublinear in } P\text{)} \\ L(P) & = \sqrt{\ln P}/4 & \text{(sublogarithmic in } P\text{)} \\ \varepsilon(P) & = P^{-1/16} & \text{(sublinear in } P \text{ decay to zero, technically largest)} \end{array}$$

By the previous lemma, there is P_0 such that for every $P \ge P_0$,

$$\mu A(P) \ge 1 - \frac{1}{P(P-1)}.$$

Let k_0 be such that $\mu A(k_0) \ge 1 - \frac{1}{k_0(k_0-1)}$. Define for every $k \ge k_0$,

$E_{c(k,0)} = (0,1),$ and for $n \ge 1$,

$E_{c(k,n)} = A(k+n) \cap E_{c(k,n-1)} \cap (\beta_n, 1),$ where $(\beta_n, 1)$ is such that

$$\mu E_{c(k,n)} = 1 - \frac{1}{k} + \frac{1}{k+n}.$$

Proposition

Let $k \geq k_0$*. Then,* $\bigcap_{n \geq 1} E_{c(k,n)}$ *consists entirely of normal numbers.*

Proof.

Assume $x \in \bigcap_{n \geq 0} E_{c(k,n)}$. Hence, $x \in \bigcap_{n \geq 1} A(k+n)$, where

$$A(k+n) = \bigcap_{2 \leq b \leq B(k+n)} \; \bigcap_{1 \leq \ell \leq L(k+n)} \; \bigcap_{w \in \{0,\ldots,b-1\}^{L(k+n)}} \Big((0,1) \setminus Bad(\varepsilon(k+n), w, b, k+n) \Big)$$

$B(k+n)$	(sublinear in n)
$L(k+n)$	(sublogarithmic in n)
$\varepsilon(k+n)$	(sublinear in n decay to zero)

For every $\delta > 0, b \geq 2, \ell \geq 1$ exists n_0 such that
$b \leq B(k+n_0)$, $\ell \leq L(k+n_0)$ and $\delta \geq \varepsilon(k+n_0)$.
So, for every $n \geq n_0$, every $w \in \{0, \ldots b^\ell\}$,

$$x \notin Bad(\delta, w, b, k+n).$$

Thus, x is normal to each base b. $\square$

By construction, for every $k \geq k_0$ and n,

$$E_{c(k,n)} \subseteq E_{c(k,n-1)}, \qquad \mu E_{c(k,n)} > 1 - \frac{1}{k},$$

and

$$E(k) = \bigcap_n E_{c(k,n)}, \qquad \mu(E(k)) = 1 - \frac{1}{k}.$$

and consists entirely of normal numbers.
The proof of Theorem 1 is complete. $\square$

Turing's Theorem 2

<u>Theorem 2</u>

There is a rule whereby given an integer K and an infinite sequence of figures 0 and 1 (the p th figure in the sequence being $\vartheta(p)$) we can find a normal number $\alpha(K,\vartheta)$ in the interval (0,1) and in such a way that for fixed K these numbers form a set of measure at least $1-2/K$, and so that the first n figures of ϑ determine $\alpha(K,\vartheta)$ to within 2^{-n}.

Theorem (Turing's Theorem 2)

There is an algorithm that, given an integer k and an infinite sequence ν of zeros and ones, produces a normal number $\alpha(k, \nu)$ in the unit interval, expressed in base two.

In order to write down the first n digits of $\alpha(k, \nu)$ the algorithm requires at most the first n digits of ν.

For a fixed k these numbers $\alpha(k, \nu)$ form a set of measure at least $1 - 2/k$.

A real number x is computable if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is the n-the digit in the expansion of x in some base. Equivalently, a real number is computable if there is a Turing machine, hence and algorithm, that outputs all of its digits, one after the other.

Proof of Turing's Theorem 2

It works by steps.

Start with the unit interval, $I_0 = (0, 1)$

At each step n, divide the current interval I_{n-1} in two halves, I_n^0, I_n^1 and choose the half that includes normal numbers in large-enough measure,

$$\mu\big(E_{c(k,n)} \cap I_n^0\big) > \text{threshold ?} \qquad \mu\big(E_{c(k,n)} \cap I_n^1\big) > \text{threshold ?}$$

If both halves do, use the current bit of the oracle to decide I_n.
(this will happen infinitely often)

The output $\alpha(k, \nu)$ is the trace of the left/right selection at each step.

Turing's Algorithm

4

With each integer n we associate an interval of the form $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^n}\right)$ whose intersection with $E_{(K)}$ is of positive measure. and given m_n we obtain m_{n+1} as follows. Put

$$m\, E_{c(K,n)} \cap \left(\frac{m_n}{2^n}, \frac{2m_n+1}{2^{n+1}}\right) = a_{n,m}$$

$$m\, E_{c(K,n)} \cap \left(\frac{2m_n+1}{2^{n+1}}, \frac{m_n+1}{2^n}\right) = b_{n,m}$$

and let r_n be the smallest m for which either $a_{n,m} < K^{-1}2^{-2n}$ or $b_{n,m} < K^{-1}2^{-2n}$ or both $a_{n,m} > \frac{1}{K(K+n+1)}$ and $b_{n,m} > \frac{1}{K(K+n+1)}$. There exists such an r_n for $a_{n,m}$ and $b_{n,m}$ decrease either to 0 or to some positive number. In the case where $a_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n+1$: if $a_{n,r_n} \geqslant K^{-1}2^{-2n}$ but $b_{n,r_n} < K^{-1}2^{-2n}$ we put $m_{n+1} = 2m_n$, and in the third case we put $m_{n+1} = 2m_n$ or $m_{n+1} = 2m_n+1$ according as $\vartheta(n) = 0$ or 1. For each n the interval $\left(\frac{m_n}{2^n}, \frac{m_n+1}{2^{n+1}}\right)$ includes normal numbers in positive measure. The intersection of these intervals contains only one number which must be normal.

Proof of Turing's Theorem 2

We need the sets $E_{d(k,n)}$ instead of $E_{c(k,n)}$.
They are defined in terms of $A(k2^{2n+1})$, instead of $A(k+n)$, as follows.

Fix k_0 and define $d(k,n)$ of two integer values in pairs of rational numbers such that for every $k \geq k_0$,

$$E_{d(k,0)} = (0,1),$$

$$E_{d(k,n)} = A(k2^{2n+1}) \cap E_{d(k,n-1)} \cap (\beta_n, 1), \text{ where } (\beta_n, 1) \text{ is such that}$$

$$\mu E_{d(k,n)} = 1 - \frac{1}{k} + \frac{1}{k2^{2n+1}}.$$

Notice $\mu\big(\bigcap_{n\geq 0} E_{d(k,n)}\big) = 1 - \frac{1}{k}$.

Proof of Turing's Theorem 2

Algorithm

Input. $k \geq k_0$, $\nu \in \{0,1\}^{\mathbb{N}}$.

Output. The unique $\alpha(k,\nu) \in \bigcap_{n \geq 1} I_n$.

Step 0*:* $I_0 = (0,1)$.

Step $n > 0$*:*

Divide I_{n-1} in two halves: I_n^0 and I_n^1.

If $\left(\mu\left(E_{d(k,n)} \cap I_n^0\right) > \frac{1}{k2^{2n}} \quad \text{and} \quad \mu\left(E_{d(k,n)} \cap I_n^1\right) > \frac{1}{k2^{2n}}\right)$ then

$I_n = I_n^{\nu(n)}$.

Else if $\left(\mu\left(E_{d(k,n)} \cap (I_n^0)\right) > \frac{1}{k2^{2n}}\right)$ then

$I_n = I_n^0$.

Otherwise $I_n = I_n^1$.

Proof of Turing's Theorem 2

Proposition

For every $n \geq 0$*,* $\mu(E_{d(k,n)} \cap I_n) > \dfrac{1}{k2^{2n}}$*.*

Proof. By induction. IH $\quad \mu(E_{d(k,n)} \cap I_n) > \dfrac{1}{k2^{2n}}$.

Case $n = 0$. $E_{d(k,0)} = (0,1)$ and $\mu E_{d(k,0)} = 1$.

Case $n > 0$. The candidates at step $n+1$ in I_n are exactly
(the candidates at step n in I_n) minus
(the candidates at step n that are not candidates step $(n+1)$, in I_n):

$$E_{d(k,n+1)} \cap I_n = \big(E_{d(k,n)} \cap I_n\big) \ \setminus \ \Big(\big(E_{d(k,n)} \setminus E_{d(k,n+1)}\big) \cap I_n\Big)$$

Then,

$$\begin{aligned}\mu\big(E_{d(k,n+1)} \cap I_n\big) &= \mu\big(E_{d(k,n)} \cap I_n\big) - \mu\Big(\big(E_{d(k,n)} \setminus E_{d(k,n+1)}\big) \cap I_n\Big)\\ &\geq \mu\big(E_{d(k,n)} \cap I_n\big) - \mu\big(E_{d(k,n)} \setminus E_{d(k,n+1)}\big).\end{aligned}$$

Using IH and $\mu E_{d(k,n)} = 1 - \dfrac{1}{k} + \dfrac{1}{k2^{2n+1}}$,

$$\mu\big(E_{d(k,n+1)} \cap I_n\big) > \frac{1}{k2^{2n}} - \left(\frac{1}{k2^{2n+1}} - \frac{1}{k2^{2n+3}}\right) > \boxed{\frac{2}{k2^{2(n+1)}}}$$

Since $I_n = I^0_{n+1} \cup I^1_{n+1}$, it is impossible that both $\mu\big(E_{d(k,n+1)} \cap I^0_{n+1}\big)$ and $\mu\big(E_{d(k,n+1)} \cap I^1_{n+1}\big)$ be less than or equal to $1/k2^{2(n+1)}$.
At least one is greater. $\square$

Proof of Turing's Theorem 2

Proposition

Let $k \geq k_0$. Then, $\bigcap_{n \geq 1} E_{d(k,n)}$ *consists entirely of normal numbers.*

Lemma (Piatetski-Shapiro 1957)

A real x is normal to an integer base $b \geq 2$, if and only if, there is a constant C such that for infinitely many lengths ℓ and for every $w \in \{0, \ldots b-1\}^{\ell}$,

$$\limsup_{P \to \infty} \frac{|x_b \upharpoonright P|_w}{P} < C \cdot b^{-\ell}.$$

Rediscovered by Borwein and Bailey 2008 calling it Hot Spot Lemma.

Proof of Proposition $\bigcap_{n\geq 0} E_{d(k,n)}$ consists entirely of normal numbers

Let $P_n = k2^{2n+1}$. Then,

$$E_{d(k,n)} = A(P_n) \cap E_{d(k,n-1)} \cap (\beta_n, 1).$$

where

$$A(P_n) = \bigcap_{2\leq b\leq B(P_n)} \bigcap_{1\leq \ell\leq L(P_n)} \bigcap_{w\in\{0,\ldots,b-1\}^{L(P_n)}} \Big((0,1)\backslash Bad(\varepsilon(P_n), w, b, P_n)\Big).$$

Then, if $x \in \bigcap_{n\geq 0} E_{d(k,n)}$ then $\forall n \geq 0, \left|\frac{|x_b \upharpoonright P_n|_w|}{P_n} - b^{-\ell}\right| \leq \varepsilon(P_n)$.

But we need it for all positions P, not just for all P_n.

Suppose $x \in \bigcap_{n \geq 0} E_{d(k,n)}$. Let M be an arbitrary position. Let n be such

$$P_n \leq M < P_{n+1}$$

Using $P_{n+1}/P_n = 4$, and

$$\frac{|x_b \restriction M|_w}{M} \leq \frac{|x_b \restriction P_{n+1}|_w}{P_n} < \frac{P_{n+1}}{P_n}\left(b^{-\ell} + \varepsilon(P_{n+1})\right) = 4\left(b^{-\ell} + \varepsilon(P_{n+1})\right)$$

As n increases, P_n goes to ∞ and $\varepsilon(P_n)$ goes to 0. Then,

$$\limsup_{P \to \infty} \frac{|x_b \restriction P|_w}{P} < 4b^{-|w|}.$$

By Piatetski-Shapiro theorem x is normal to each base b. □

Proof of Turing's Theorem 2

Proposition

The number $\alpha(k, \nu)$ output by the algorithm is normal.

Proof.

Since $(I_n)_{n \geq 0}$ is a nested sequence of intervals and $\mu\Big(E_{d(k,n)} \cap I_n\Big) > 0$, for every n, we obtain that

$$\bigcap_{n \geq 0} I_n = \bigcap_{n \geq 0} \big(E_{d(k,n)} \cap I_n\big).$$

Since already showed that $\tilde{E}(k) = \bigcap_{n \geq 0} E_{d(k,n)}$ consists just of normal numbers, then $\bigcap_{n \geq 0}(E_{d(k,n)} \cap I_n)$ contains a unique real number $\alpha(k, \nu)$ that is normal,. $\square$

Proof of Turing's Theorem 2

Proposition

The set of real numbers produced by the algorithm by varying $\nu \in \{0,1\}^{\mathbb{N}}$ has measure at least $1 - 2/k$.

Proof. For each $n \geq 1$,$m = 0, \ldots 2^n - 1$. let $J_{n,m} = \left(\frac{m}{2^n}, \frac{m+1}{2^n}\right)$.

Define inductively the set $M(k, n)$ consisting of all possible intervals $J_{n,m}$ as we allow the first n digits of ν to run through all possibilities. That is, having deleted those intervals that would be discarded by the algorithm up to step $n - 1$.

Define $M : \mathbb{N} \times \mathbb{N} \to \mathcal{P}(0, 1)$,
$M(k, 0) = (0, 1)$, and for $n > 0$,

$$M(k, n) = \bigcup_{\substack{J_{n,m} \subseteq M(k,n-1) \\ \mu\left(E_{d(k,n)} \cap J_{n,m}\right) > 1/k2^{2n}}} J_{n,m}$$

An interval $J_{n,m}$ is descarded at level n if it was possible at level $n - 1$ but it is not possible at level n, (because it fails the threshold at level n).
Let $\widetilde{E}(k) = \bigcap_{n \geq 0} E_{d(k,n)}$. For $m = 0, 1, \ldots 2^n - 1$,

$$D_{n,m} = \widetilde{E}(k) \cap J_{n,m} \cap (M(k, n-1) \setminus M(k, n))$$

Thus, $\mu D_{n,m} \leq \dfrac{1}{k2^{2n}}$.
Want $\mu\Big(\widetilde{E}(k) \cap \bigcap_n M(k, n)\Big)$

$$\widetilde{E}(k) \cap M(k,n) = \widetilde{E}(k) \cap M(k,n-1) \setminus \bigcup_{0 \le m < 2^n} D_{n,m}$$

$$= \widetilde{E}(k) \cap M(k,n-2) \setminus \Big(\bigcup_{0 \le m < 2^{n-1}} D_{n-1,m} \cup \bigcup_{0 \le m < 2^n} D_{n,m} \Big)$$

$$\ldots$$

$$= \widetilde{E}(k) \cap M(k,0) \setminus \Big(\bigcup_{1 \le j \le n} \bigcup_{0 \le m < 2^j} D_{j,m} \Big)$$

Since $M(k,0) = (0,1)$, $\mu\widetilde{E}(k) = 1 - \frac{1}{k}$, and $\mu D_{j,m} \le \frac{1}{k2^{2j}}$,

$$\mu(\widetilde{E}(k) \cap M(k,n)) = \mu(\widetilde{E}(k) \cap M(k,0)) - \Big(\sum_{1 \le j \le n} \sum_{0 \le m < 2^j} \mu D_{j,m} \Big)$$

$$\ge \mu\widetilde{E}(k) - \Big(\sum_{1 \le j \le n} \sum_{0 \le m < 2^j} \frac{1}{k2^{2j}} \Big)$$

$$> \Big(1 - \frac{1}{k}\Big) - \frac{1}{k}$$

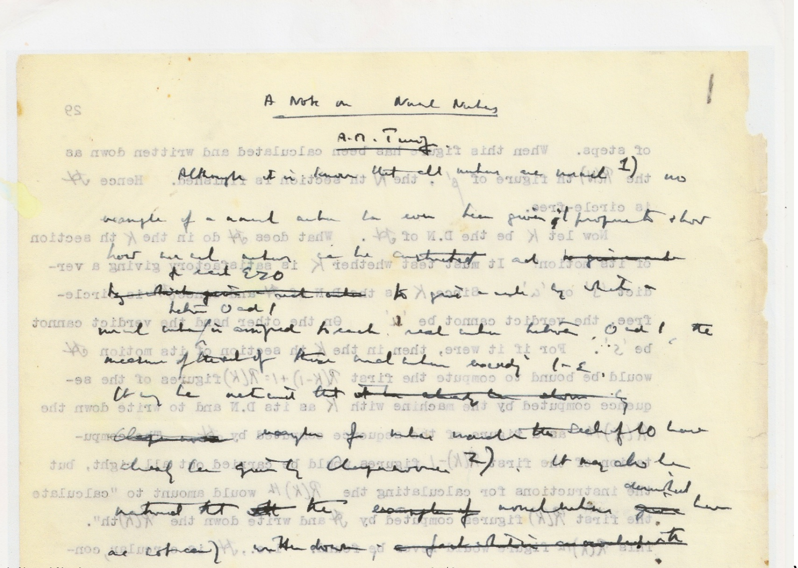
$$= 1 - \frac{2}{k}.$$

We conclude $\mu\Big(\widetilde{E}(k) \cap \bigcap_n M(k,n)\Big) \ge 1 - \frac{2}{k}$. $\square$

The proof of Turing's Theorem 2 is now complete.

Turing's First Page of the Handwritten Manuscript

Not transcribed. His own appraisal of his work.



A Note on Normal Numbers

A. M. Turing

Turing's First Page of the Handwritten Manuscript

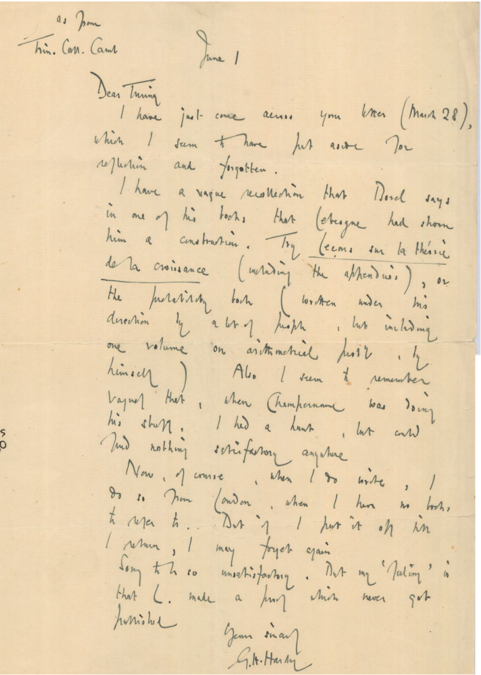
"No example of a normal number has ever been given."

Turing cites Champernowne's $0.12345678910111213141 5\ldots$
as an example of a number that is normal to base ten.

"It may also be natural that an example of a normal number be demonstrated as such and written down. This note cannot, therefore, be considered as providing <u>convenient</u> examples of normal numbers"//

//"but rather, to counter [...] that the existence proof of normal numbers provides no example of them. The arguments in the note, in fact, follow the existence proof fairly closely."
He was aware of the algorithm's computational complexity.

Letter G.H.Hardy to Turing (AMT/D/5)



as from
Trin. Coll. Camb

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the probability book (written under his direction by a lot of people, but including one volume on arithmetical prob, by himself) Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till my return, I may forget again

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published

Yours sincerely

G.H. Hardy

June 1

Dear Turing,

I have just came across your letter (March 28) which I seem to have put aside for reflection and forgotten.

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Sorry to be so unsatisfactory. But my 'feeling' is that Lebesgue made a proof which never got published.
Yours sincerely,

G.H. Hardy

SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q (1) tout nombre réel x dont la partie fractionnaire

(1) E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

independently, each gave a non-finitary based construction:
Bulletin de la Société Mathématique de France 45, 1917,
respectively in page s 127–132 and 132–144.

Algorithms generating absolutely normal numbers

(Double) exponential time, Turing's algorithm, 1937

Exponential time, Weyl criterion and exponential sums
Schmidt 1961/1962; Levin 1979

Nearly quadratic time algorithm
Becher, Heiber and Slaman 2013, Figueira and Nies 2013, Lutz and Mayordomo 2013,
Nearly linear time algorithm
Lutz and Mayordomo, 2021

Polynomial time, with discrepancy smaller than that almost all numbers
Aistleitner, Becher, Scheerer, Slaman 2017

Polynomial time, combining normality with continued fraction normality, a number and its reciprocal,normal numbers in Cantor sets,

Work to do

Give the polynomial counterparts of known exponential algorithms producing normal numbers: Liouville, prescribed irrationality exponent, Toeplitz numbers.

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