$\mathcal{N}$ormalℓ numbers, L$o$gic and $\mathcal{A}$utomata

Verónica Becher

Universidad de Buenos Aires & CONICET, Argentina

Logic Colloquium 2017, Stockholm University
August 14-20, 2017

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Special highlight session LC-CSL

Normal numbers

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I will present some results obtained with tools of
Logic and Automata theory.

Representation of real numbers

A *base* is an integer b greater than or equal to 2.

The expansion of a real number x in base b is a sequence $a_1 a_2 a_3 \ldots$ of integers from $\{0, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k \geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

and the sequence $a_1 a_2 a_3 \ldots$ does not end with a tail of $b-1$.

Normal numbers

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A real x is absolutely normal if x is normal to every base.
Hence, a real x is absolutely normal if it is simply normal to all bases b.

Examples and counterexamples

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Each number that is simply normal to b^k is simply normal to base b.

$0.123456789101112131415\ldots$ is normal to base 10 (Champernowne, 1933).
It is unknown if it simply normal to bases that are not powers of 10.

Absolutely normal numbers

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The set of absolutely normal numbers in the unit interval has Lebesgue measure one.

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Borel asked for an explicit example .

1

Exhibit an absolutely normal number

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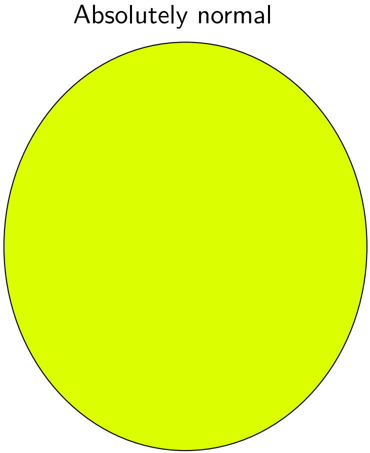

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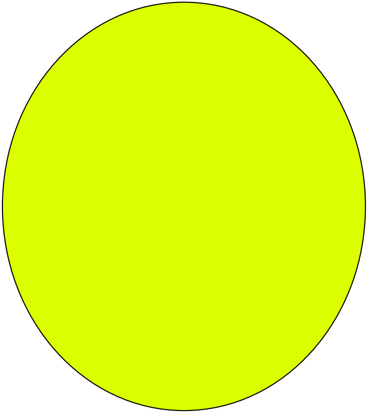

π? $\sqrt{2}$?

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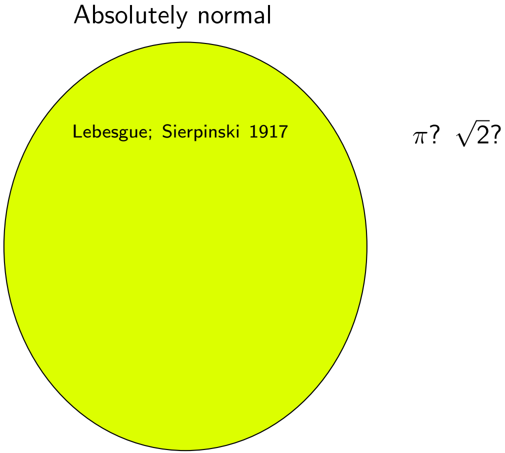

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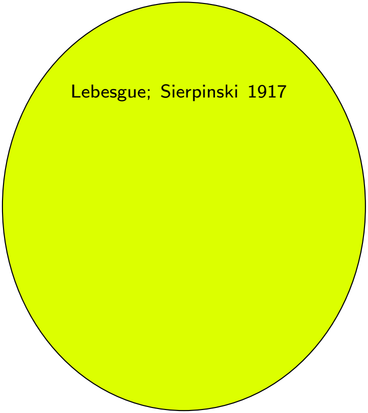

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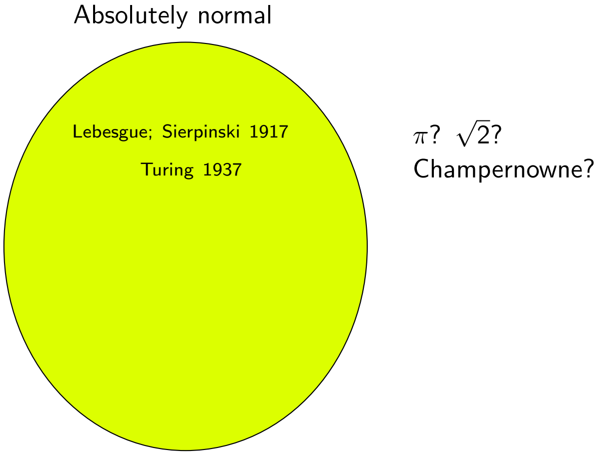

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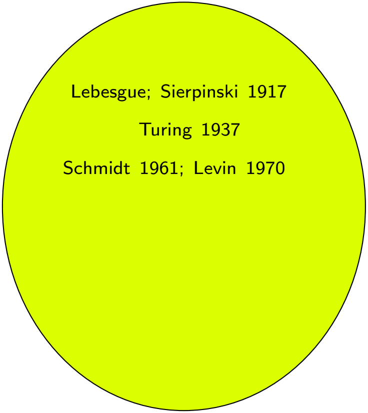

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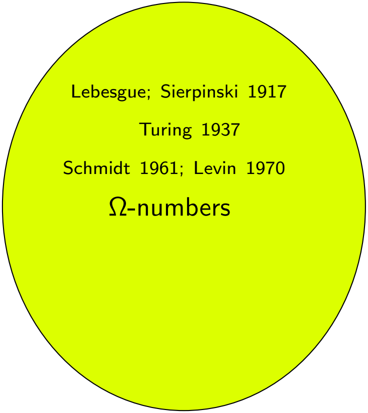

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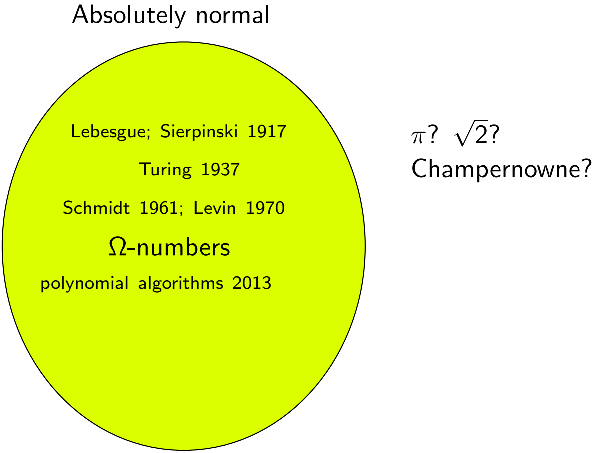

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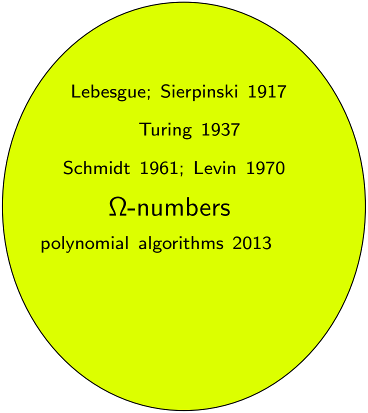

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Lutz and Mayordomo 2013; Figuiera and Nies 2013

Becher, Heiber and Slaman 2013: 0.4031290542003809132371428380827059102765116777624189777

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Absolutely normal

Lebesgue; Sierpinski 1917

Turing 1937

Schmidt 1961; Levin 1970

Ω-numbers

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quasilinear, Lutz and Mayordomo 2016

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Other properties (Liouville, fast convergence to normality).

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A real number x is computable is there is a program that produces the expansion of x in some base.

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Theorem (Turing 1937?)

There is a computable absolutely normal number.

Turing inductively defines a set that contains all non normal numbers, and at the same time he inductively defines a number x outside this set.

To produce the n-th binary digit of x, Turing's algorithm performs a number of operations that is exponential in n.

1. Exhibit an absolutely normal number

Conjecture (Borel 1951)

Irrational algebraic numbers are absolutely normal.

2 Normality to different bases

2. Normality to different bases

Two positive integers are multiplicatively dependent if one is a rational power of the other. For example 2 and 8 are multiplicatively dependent, but 2 and 6 are not.

Theorem (Maxfield 1953)

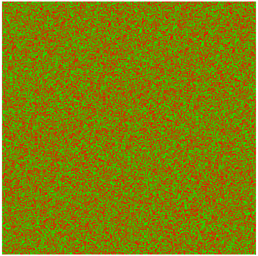
Let b_1 and b_2 multiplicatively dependent. For any real number x, x is normal to base b_1 if and only if x is normal to base b_2.

2. Normality to different bases

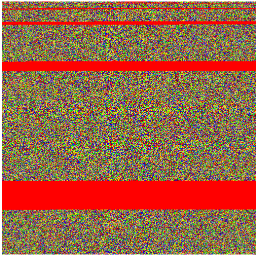
Bailey and Borwein (2012) proved that the Stoneham number $\alpha_{2,3}$,

$$\alpha_{2,3} = \sum_{k \geq 1} \frac{1}{3^k \, 2^{3^k}}$$

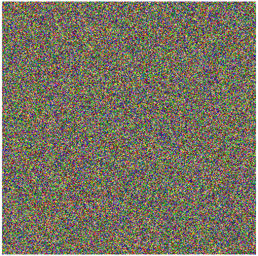
is normal to base 2 but not simply normal to base 6.



base 2



base 6



base 10

2. Normality to different bases

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Theorem (Schmidt 1961/1962)

For any given set S of bases closed under multiplicative dependence, there are real numbers normal to every base in S and not normal to any base in its complement. Furthermore, there is a real x is computable from S.

Improved by Becher and Slaman 2014 to obtain lack of simple normality.

Becher, Bugeaud and Slaman, 2016, proved the analog of this theorem for simple normality.

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Asked first by Kechris 1994,
What is the descriptive complexity of the set of normal numbers?

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Recall that the Borel hierarchy for subsets of the real numbers is the stratification of the σ-algebra generated by the open sets with the usual interval topology.

When we restrict to intervals with rational endpoints and computable countable unions and intersections, we obtain the effective Borel hierarchy.

2. Normality to different bases (Kechris's question)

A real x is simply normal to base b if

$$\forall d \in \{0, ..., b-1\} \lim_{n \to \infty} \left| \frac{|a_1 \dots a_n|_d}{n} - \frac{1}{b} \right| = 0$$

where $a_1 a_2 \dots$ is the expansion of x in base b.

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Equivalently,

$$\forall d \in \{0, .., b-1\} \ \forall \varepsilon \ \exists n_0 \ \forall n \geq n_0 \ \left| \frac{|a_1 \dots a_n|_d}{n} - \frac{1}{b} \right| < \varepsilon.$$

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$\forall\exists\forall$ yields a Π^0_3 formula over the reals.
Simple normality, normality and absolute normality are defined by Π^0_3 formula.

2. Normality to different (Kechris's question)

Theorem (Ki and Linton 1994)

For a fixed base b, the set of reals that are normal to base b is $\mathbf{\Pi}^0_3$*-complete and* Π^0_3*-complete.*

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The set absolutely normal reals is $\mathbf{\Pi}^0_3$-complete and Π^0_3-complete.

Corollary

Since the set of Martin-Löf random reals is $\mathbf{\Sigma}^0_2$-complete, it is different from the set of normal reals.

2. Normality to different bases

We confirmed a conjecture by Achim Ditzen, 1994:

Theorem (Becher and Slaman 2014)

The set of real numbers that are ***normal to at least one*** *base is* $\boldsymbol{\Sigma}^0_4$*-complete and* Σ^0_4*-complete.*

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Given a Σ^0_4 sentence we produce a real x.

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Theorem (Becher and Slaman 2014)

For each Π^0_3 formula φ in second order arithmetic there is a computable real number x such that, for any non-perfect power b, x is normal to base b if and only if $\varphi(x, b)$ is true.

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So, for each Π^0_3 formula φ there is x such that normality of x to base b has the same truth value as $\varphi(x, b)$.

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So, for each Π^0_3 formula φ there is x such that normality of x to base b has the same truth value as $\varphi(x, b)$.

There is no logical dependence between normality between different bases, other than multiplicatively dependence.

2. Another result on descriptive complexity

Theorem (Airey, Jackson and Mance, 2016)

Let N_b be the set of real numbers which are normal to a given base b. The set of real numbers that are normal to base b and preserve normality to base b under addition ,

$$\{x : x \in N_b \text{ and } \forall y \in N_b \ (x + y \in N_b)\},$$

is Π^0_3-complete.

3

Normality of infinite words: unpredictability/incompressibility by finite automata

3. Normality and finite automata

Theorem

normality *iff* *no finite-state martingale success*

(Schnorr and Stimm 1971)

3. Normality and finite automata

Theorem

normality	*iff*	*no finite-state martingale success* (Schnorr and Stimm 1971)
no finite-state martingale success	*iff*	*incompressibility* (Dai, Lathrop, Lutz and Mayordomo 2004) (Bourke, Hitchcock and Vinodchandran 2005)

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	iff	*incompressibility by non-deterministic*
	iff	*incompressibility by one counter* (Becher, Carton and Heiber 2015)

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	iff	*incompressibility by non-deterministic*
	iff	*incompressibility by one counter* (Becher, Carton and Heiber 2015)
	iff	*incompressibility two-way transducers* (Carton and Heiber 2015)

3. Normality as incompressibility by finite automata

Definition

A deterministic finite transducer is a tuple $\mathcal{A} = \langle Q, A, \delta, q_0 \rangle$, where

- Q is a finite set of *states*,
- A is the input and output alphabet
- $\delta : Q \times A \to A^* \times Q$ is a transition function, where a transition is written $p \xrightarrow{a|v} q$.
- q_0 is *initial* state.

A *run* with input $x = a_1 a_2 \ldots$ is a sequence of consecutive transitions,

$$q_0 \xrightarrow{a_1|v_1} q_1 \xrightarrow{a_2|v_2} q_2 \cdots q_{n-1} \xrightarrow{a_n|v_n} q_n \ldots$$

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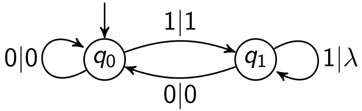
A transducer $\mathcal{A}$ is one-to-one if the function $x \mapsto \mathcal{A}(x)$ is one to one.

3. Normality as incompressibility by finite automata

Example

A finite transducer that transforms blocks of 1*s into a single* 1*.*



If $x = 010011000111...$, *then* $\mathcal{A}(x) = 01001000100...$

Beware! It is not one-to-one.

3. Normality as incompressibility by finite automata

Suppose the run in $\mathcal{A}$ with iput $x = a_1 a_2 \ldots$ is

$$q_0 \xrightarrow{a_1 | v_1} q_1 \xrightarrow{a_2 | v_2} q_2 \xrightarrow{a_3 | v_3} q_3 \cdots$$

Then, the compression ratio of $x = a_1 a_2 \ldots$ in $\mathcal{A}$ is

$$\rho_{\mathcal{A}}(x) = \liminf_{n \to \infty} \frac{|v_1 v_2 \cdots v_n|}{n}$$

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The compression ratio of $x = a_1 a_2 a_3 \cdots$ is

$$\rho(x) = \inf \{\rho_{\mathcal{A}}(x) : \mathcal{A} \text{ is one-to-one}\}$$

We say x is compressible if only if $\rho(x) < 1$.

3. Normality as incompressibility by finite automata

Problem

Is there a deterministic push-down one-to-one transducer and a normal word which is compressed by it?

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Theorem (Boasson 2014)

There is a non-deterministic push-down one-to-one transducer and a normal word which is compressed by it.

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Theorem (Boasson 2014)

There is a non-deterministic push-down one-to-one transducer and a normal word which is compressed by it.

Proof.

0123456789 9876543210 00010203 ... 979899 999897...03020100 . . . □

4

Independence of normal words

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When are two normal words independent?

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First attempt of a definition of independence:
Two normal words are independent exactly when their join is normal.

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Theorem (Becher, Carton and Heiber 2016)

There are two normal words x and y such that x join $y = x$.

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Two normal words are independent exactly when their join is normal.

Theorem (Becher, Carton and Heiber 2016)

There are two normal words x and y such that x join $y = x$.

Here $x = even(x)$ and $y = odd(x)$, hence they are obviously dependent.

Theorem (Shen 2016)

Let $x_1, x_3, x_5, \ldots$ be uniformly distributed independent symbols and for every odd n, let $x_n = x_{2n} = x_{4n} = \ldots$. Then, with probability 1 the resulting word $x_1 x_2 x_3 \ldots$ is normal.

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Definition

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- Q is the finite state set,
- A is the alphabet,
- $\delta : Q \times (A \cup \{\lambda\}) \times (A \cup \{\lambda\}) \to A^* \times Q$ is the transition function where a transition is written $p \xrightarrow{\alpha,\beta|\gamma} q$,
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- $\delta : Q \times (A \cup \{\lambda\}) \times (A \cup \{\lambda\}) \to A^* \times Q$ is the transition function where a transition is written $p \xrightarrow{\alpha,\beta|\gamma} q$,
- q_0 is the initial state.

A *run* with inputs x and y is a sequence of consecutive transitions

$$q_0 \xrightarrow{\alpha_1,\beta_1|\gamma_1} q_1 \xrightarrow{\alpha_2,\beta_2|\gamma_2} q_2 \cdots$$

4. Independence of normal numbers

Two normal words are independent exactly when one does not help to compress the other.

Definition

A deterministic finite transducer 2 input tapes and 1 output tape is a tuple $\mathcal{A} = \langle Q, A, \delta, q_0 \rangle$, where

- Q is the finite state set,
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- $\delta : Q \times (A \cup \{\lambda\}) \times (A \cup \{\lambda\}) \to A^* \times Q$ is the transition function where a transition is written $p \xrightarrow{\alpha,\beta|\gamma} q$,
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A *run* with inputs x and y is a sequence of consecutive transitions

$$q_0 \xrightarrow{\alpha_1,\beta_1|\gamma_1} q_1 \xrightarrow{\alpha_2,\beta_2|\gamma_2} q_2 \cdots$$

We write $\mathcal{A}(x, y) = \gamma_1\gamma_2\gamma_3 \cdots$.
We say $\mathcal{A}$ is one-to-one if for each y fixed, $x \to \mathcal{A}(x, y)$ is one-to-one.

4. Independence of normal numbers

Definition

Let $\mathcal{A}$ be a finite transducer with two input tapes, deterministic and one-to-one. Suppose inputs x and y and the run in $\mathcal{A}$

$$q_0 \xrightarrow{\alpha_1,\beta_2|\gamma_1} q_1 \xrightarrow{\alpha_2,\beta_2|\gamma_2} q_2 \xrightarrow{\alpha_3,\beta_3|\gamma_3} q_3 \cdots$$

where $x = \alpha_1\alpha_2\ldots$ and $y = \beta_1\beta_2\ldots$

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where $x = \alpha_1\alpha_2\ldots$ and $y = \beta_1\beta_2\ldots$

The conditional compression ratio of x with respect to y in $\mathcal{A}$ is

$$\rho_{\mathcal{A}}(x/y) = \liminf_{n\to\infty} \frac{|\gamma_1\ldots\gamma_n|}{|\alpha_1\ldots\alpha_n|}.$$

Notice that the number of symbols read from y, namely $|\beta_1\ldots\beta_n|$, is not taken into account in the value of $\rho_{\mathcal{A}}(x/y)$.

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Notice that the number of symbols read from y, namely $|\beta_1\ldots\beta_n|$, is not taken into account in the value of $\rho_{\mathcal{A}}(x/y)$.

The conditional compression ratio of x given y, $\rho(x/y)$, is the infimum of $\rho_{\mathcal{A}}(x/y)$ for all $\mathcal{A}$ deterministic one-to-one.

4. Independence of normal numbers

Definition

Two words x and y are independent if their compression ratios are not 0 and y does not help to compress x and x does not help to compress y,

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Two words x and y are independent if their compression ratios are not 0 and y does not help to compress x and x does not help to compress y,

$$\rho(x) = \rho(x/y) > 0 \text{ and } \rho(y) = \rho(y/x) > 0.$$

4. Independence of normal numbers

Theorem (Becher and Carton 2016)

The set $\{(x, y) : x$ *and* y *are independent*$\}$ *has Lebesgue measure* 1.

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Lemma

The set of words that are compressible with the help of a given normal word has Lebesgue measure 0.

4. Independence of normal numbers

Definition

A shuffler $\mathcal{S} = \langle Q, A, \delta, q_0 \rangle$ is a finite transducer with two input tapes and one output tape. The transition function is $\delta : Q \times A \cup \{\lambda\} \times A \cup \{\lambda\} \to Q \times A$, transitions have the form

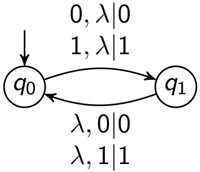
$$p \xrightarrow{a,\lambda|a} q \quad \text{or} \quad p \xrightarrow{\lambda,a|a} q.$$

For each state q, all incoming transitions have the same type.

Whether the next digit is taken from the first or the second input word only depends the current state.

4. Independence of normal numbers

Example of a Shuffler that computes the join



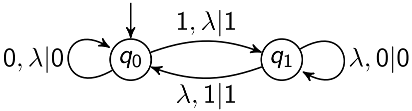
$x = 0011010001...$

$y = 0100011000...$

x join $y = 00011010001101000010...$

4. Independence of normal numbers

Example of another shuffler. It alternates (possibly empty) blocks of 0s followed by a 1, from each input word.



Input words $\begin{cases} x = 001\,1\,01\,0001\ldots \\ y = 01\,0001\,1\,0001\ldots \end{cases}$

Output word $z = 0010110001011000 10001\ldots$

4. Independence of normal numbers

Theorem (Alvarez, Becher and Carton 2016)

Two normal words are independent if and only if every shuffling is normal.

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Theorem (Alvarez, Becher and Carton 2016)

There is an algorithm that computes two normal independent words.

4. Independence of normal numbers

Problem

Give combinatorial characterization of finite-state independence.

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Give combinatorial characterization of finite-state independence.

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Construct $x = a_1 a_2 \ldots$ *normal such that for all* n, $a_{2n} = a_n$ *and* $a_{3n} = a_n$.

4. Independence of normal numbers

Problem

Give combinatorial characterization of finite-state independence.

Problem

Construct $x = a_1 a_2 \ldots$ *normal such that for all* n, $a_{2n} = a_n$ *and* $a_{3n} = a_n$.

Problem

Construct a normal word that is independent of Champernowne.

Concluding remark

Little is known about the interplay between combinatorial, computational and number theoretic properties of real numbers.

These investigations on normal numbers aim to make progress in this direction.

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Little is known about the interplay between combinatorial, computational and number theoretic properties of real numbers.

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The End

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