On normal numbers

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Normal numbers

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Definition

A base is an integer b greater than or equal to 2. For a real number x, the expansion of x in base b is a sequence $(a_k)_{k\geq 1}$ of integers a_k from $\{0, 1, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k\geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1 a_2 a_3 \ldots$$

where infinitely many of the a_k are not equal to $b-1$.

Normal numbers

Definition (Borel, 1909)

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A real number x is normal to base b if x is simply normal to every base b^k, for every positive integer k.

Normal numbers

Theorem (Borel 1922, Niven and Zuckerman 1951)

A real number x is normal to base b if, for every $k \geq 1$, every block of k digits occurs in the expansion of x in base b with limiting frequency $1/b^k$.

Normality as uniform distribution modulo one

Theorem (Wall 1949)

A real x is normal to base b if and only if $(b^k x)_{k\geq 0}$ equidistributes modulo one for Lebesgue measure.

Not normal

$0.01\ 002\ 0003\ 00004\ 000005\ 0000006\ 00000007\ 000000008\ldots$
is not simply normal to base 10.

Not normal

$0.01\,002\,0003\,00004\,000005\,0000006\,00000007\,000000008\ldots$
is not simply normal to base 10.

$0.0123456789\ \ 0123456789\ \ 0123456789\ \ 0123456789\ \ 0123456789\ldots$
is simply normal to base 10, but not simply normal to base 100.

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$0.0123456789\ \ 0123456789\ \ 0123456789\ \ 0123456789\ \ 0123456789\ldots$
is simply normal to base 10, but not simply normal to base 100.

Rational numbers are normal to no base.

Normal to a given base

Theorem (Champernowne, 1933)

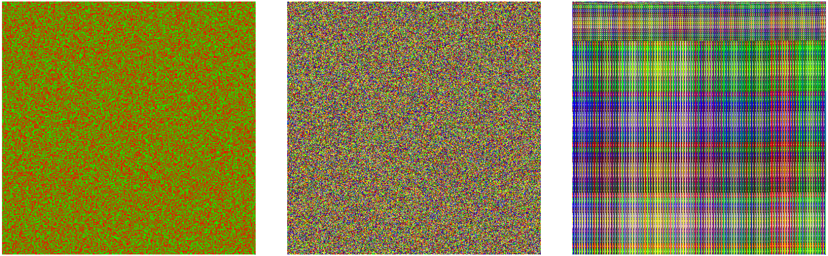
$0.12345678910111213141516171819202122232425\ldots$ *is normal to base* 10.

Normal to a given base

Theorem (Champernowne, 1933)

$0.12345678910111213141516171819202122232425\ldots$ *is normal to base* 10.

It is unknown if it is normal to bases that are not powers of 10.



base 2 base 6 base 10

Plots of the first 250000 digits of Champernowne's number.

Normal to all bases

Theorem (Borel 1909)

Almost all real numbers are absolutely normal.

Problem (Borel 1909)

Give one example.

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Problem (Borel, 1909)

Are the usual mathematical constants, such as π, e, or $\sqrt{2}$, absolutely normal?
Or at least simply normal to some base?

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Constructions can not be just concatenation

Concatenation works if we consider just one base.
For two bases, concatenation in general fails.

For example,

	base 10	*base* 3
$x =$	$(0.25)_{10} =$	$(0.020202020202\ldots)_3$...
$y =$	$(0.0017)_{10} =$	$(0.000001020110110010 2\ldots)_3$
$x + y =$	$(0.2517)_{10} =$	$(0.0202101110122\ldots)_3$

Normal to all bases

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base q (1) tout nombre réel x dont la partie fractionnaire

(1) E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;

PAR M. H. LEBESGUE.

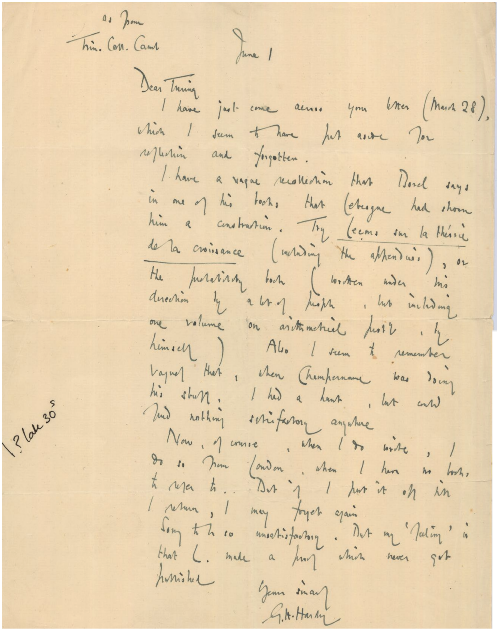
Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

Normal to all bases

First ones by Lebesgue and independently Sierpiński in 1917 (not computable).
Several constructions 1917–2013, including Turing 1937, Schmidt 1961/2.

Recall that a real number is computable if there is an algorithm that computes its expansion in some base (given n the algorithm computes the n-th digit).

Normal to all bases



as from
Trin. Coll. Camb.

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the probability book (written under his direction by a lot of people, but including one volume on arithmetical parts, by himself) Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere.

Now, of course, when I do write, I do so from London, where I have no books to refer to. But if I put it off till I return, I may forget again.

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published.

Yours sincerely
G.H. Hardy

? late 30s

Normal to all bases

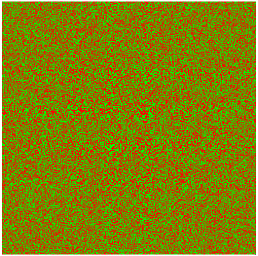
Theorem (Becher, Heiber and Slaman, 2013)

There is an algorithm that computes an absolutely normal number with just above quadratic time-complexity.

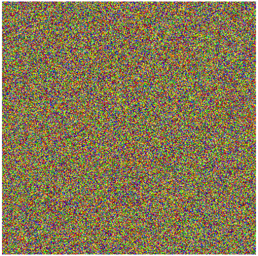
Normal to all bases

Output of algorithm Becher, Heiber and Slaman, 2013 programmed by Martin Epszteyn.

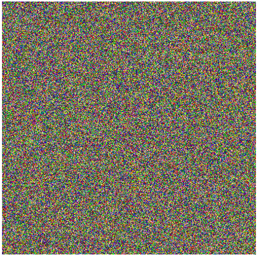
$0.4031290542003809132371428380827059102765116777624189775110896366\ldots$



base 2



base 6



base10

Plots of the first 250000 digits of the output of our algorithm.

Also in 2013 polynomial-time algorithms by Lutz and Mayordomo and by Figueira and Nies.

Normality to different bases

Definition

Two positive integers are multiplicatively dependent if one is a rational power of the other. Then, 2 and 8 are dependent, but 2 and 6 are independent.

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The positive integers that are not perfect powers, $2, 3, 5, 6, 7, 10, 11, \ldots$, are pairwise multiplicatively independent.

Theorem (Maxfield 1953)

Let b_1 and b_2 multiplicatively dependent. For any real number x, x is normal to base b_1 if and only if x is normal to base b_2.

Normality to different bases

Theorem (Cassels, 1959)

Almost every real number in the middle third Cantor set is normal to every base which is not a power of 3.

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Theorem (Schmidt 1961/1962)

For any given set S of bases closed under multiplicative dependence, there are real numbers normal to every base in S and not normal to any base in its complement. Furthermore, there is a real x computable from S.

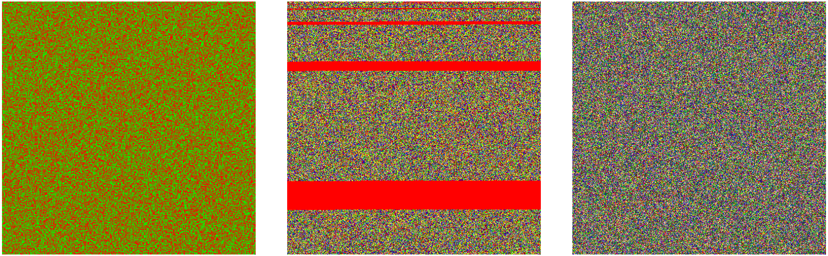
Becher and Slaman 2014 refuted simple normality answering to Brown, Moran and Pearce.

An example with a closed formula

Bailey and Borwein (2012) proved that the Stoneham number $\alpha_{2,3}$,

$$\alpha_{2,3} = \sum_{k \geq 1} \frac{1}{3^k \; 2^{3^k}}$$

is normal to base 2 but not simply normal to base 6.



base 2 base 6 base 10

Plots of the first 250000 digits of Stoneham number $\alpha_{2,3}$.

Arithmetical independence

Theorem (informal statement) (Becher and Slaman 2014)

The set of bases to which a real number can be normal is not tied to any arithmetical properties other than multiplicative dependence.

Arithmetical independence

Formally, this is a problem in descriptive set theory. Achim Ditzen conjectured it in 1994 (after a question of A. Kechris). We confirmed it.

Theorem (Becher and Slaman 2014)

The set of real numbers that are normal to some base is complete at the fourth level of the Borel Hierachy on subsets of real numbers.

Arithmetical independence

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The proof shows that the discrepancy functions (speed of convergence to normality) on multiplicatively independent bases are pairwise independent.

An old silent question

Made explicit by Yann Bugeaud in 2013

What are the necessary and sufficient conditions on a set of bases so that there is a real number which is simply normal exactly to the bases in such a set?

Simple normality to different bases

Simple normality to base 8 implies simple normality to base 2, because $8 = 2^3$.

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Simple normality to infinitely many powers of b implies normality to b (Long 1957)

Necessary and sufficient conditions for simple normality

Theorem (Becher, Bugeaud and Slaman 2013)

Let f be any function from the set of integers that are not perfect powers to sets of integers such that for each b,

- *if for some k, b^k is in $f(b)$ then, for every ℓ that divides k, b^ℓ is in $f(b)$;*
- *if $f(b)$ is infinite then $f(b) = \{b^k : k \geq 1\}$.*

Then, there is a real x simply normal to exactly the bases specified by f. Moreover, the set of real numbers that satisfy this condition has full Hausdorff dimension.

Also, the real x is computable from the function f.

New normal numbers

Belief

If we consider appropriate measures, most elements of well structured sets are absolutely normal, unless the sets have evident obstacles.

Normality and Weyl's criterion

A number x is normal to base b if $(b^k x)_{k\geq 0}$ is uniformly distributed modulo one.

A sequence of real numbers is uniformly distributed if and only if for every Riemann-integrable (complex-valued) function f, $\int_0^1 f(x)dx$ is the limit of the average values of f on the sequence.

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By Weyl's criterion, a real number x is normal to base b if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} = 0.$$

Appropriate measures for normality

Lemma (direct application of Davenport, Erdős, LeVeque's Theorem 1963)

Let μ be a measure, I an interval and b a base. If for every non-zero integer t,

$$\sum_{n \geq 1} \frac{1}{n} \int_I \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} \right|^2 d\mu(x) \ < \infty$$

then for μ-almost all x in interval I are normal to base b.

Normal numbers and Diophantine approximations

Definition (Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which the inequality $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$ is satisfied by an infinite number of integer pairs (p, q) with positive q.

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- Almost all real numbers have irrationality exponent equal to 2.
- Every real greater than 2 is the irrationality exponent of some real.
- Irrational algebraic numbers have irrationality exponent equal to 2 (Thue - Siegel - Roth theorem 1955).
- Rational numbers have irrationality exponent equal to 1.

Fractals, measures and approximations

- Jarník (1929) and Besicovich (1934) defined a fractal for real numbers with irrationality exponent equal to a given a greater than or equal to 2.
- Kaufman (1981) defined for each real number a greater than 2, a measure on Jarník 's fractal whose Fourier transform decays quickly.
- Bluhm (2000) defined a measure supported by the Liouville numbers, whose Fourier transform decays quickly.
- We adapted the measures, tailored for effective approximations.

Normal Liouville numbers

Theorem (Bugeaud 2002)

There is an absolutely normal Liouville number.

Theorem (Becher, Heiber and Slaman 2014)

There is a computable absolutely normal Liouville number.

Simple normality and irrationality exponents

Theorem (Becher, Bugeaud and Slaman, in progress 2014)

Let S be a set of bases satisfying the conditions for simple normality.

- *There is a Liouville number x simply normal to exactly the bases in S.*
- *For every a greater than or equal to 2 there is a real x with irrationality exponent equal to a and simply normal to exactly the bases in S.*

Furthermore, x is computable from S and, for non- Liouville, also from a.

Normality and finite automata

Fix an alphabet A. Consider finite automata that input and output infinite sequences of symbols from A.

Definition

A sequence $a_1 a_2 a_3 \ldots$ is compressible by an input-output finite automata if

$$\liminf_{k \to \infty} \frac{\#\text{output symbols after reading } a_1 \ldots a_k}{k} < 1$$

Normality and finite automata

Theorem (Schnorr and Stimm 1971 + Dai, Lathrop, Lutz and Mayordomo 2004)

A real is normal to base b if, and only if, its expansion in base b is incompressible by injective input-output finite automata.

A direct proof of the above theorem Becher and Heiber, 2012.

Normality and finite automata

Theorem (Becher, Carton, Heiber 2013)

Non-deterministic bounded-to-one input-output finite automata, even if augmented with a fixed number of counters, can not compress expansions of normal numbers.

Theorem (Boasson 2012)

Non-deterministic pushdown input-output finite automata can compress expansions of normal numbers.

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Problem

Can deterministic pushdown input-output finite automata compress expansions of normal numbers?

Normality preservation and finite automata

Let $a_1 a_2 a_3 \cdots$ be an infinite sequence. Consider the infinite sequence obtained by selection of some elements

$a_1 \; a_2 \; \textcircled{a_3} \; a_4 \; a_5 \; \textcircled{a_6} \; \textcircled{a_7} \; a_8 \; a_9 \; \ldots$

Normality preservation and finite automata

Let $a_1 a_2 a_3 \cdots$ be an infinite sequence. Consider the infinite sequence obtained by selection of some elements

$$a_1 \ a_2 \ \textcircled{a_3} \ a_4 \ a_5 \ \textcircled{a_6} \ \textcircled{a_7} \ a_8 \ a_9 \ \ldots$$

Theorem (Agafonov 1968)

Prefix selection by a regular set of finite sequences preserves normality.

Theorem (Becher, Carton and Heiber 2013)

Suffix selection by a regular set of infinite sequences preserves normality.

Normality preservation and finite automata

Theorem (Becher, Carton and Heiber 2013)

Two sided selectors do not preserve normality.

Theorem (Merkle and Reimann 2006)

Neither deterministic one-counter sets nor linear sets preserve normality (these are the sets recognized by pushdown finite automata with a unary stack and by one-turn pushdown finite automata, respectively)

Problem

What is the least powerful selection that does not preserve normality?

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Normality and finite automata

Definition

A *transducer* is a tuple $T = \langle Q, A, B, \delta, q_0 \rangle$, where

- Q is a finite set of states,
- A and B are the input and output alphabets, respectively,
- $\delta : Q \times A \to B^* \times Q$ is the transition function,
- $q_0 \in Q$ is the starting state.

If $\delta(p, a) = \langle v, q \rangle$ we write $p \xrightarrow{a|v} q$.
An *infinite run* is $p_0 \xrightarrow{a_1|v_1} p_1 \xrightarrow{a_2|v_2} p_2 \xrightarrow{a_3|v_3} p_3 \cdots$ is accepting if $p_0 = q_0$.

This is the Büchi acceptance condition where all states are accepting.

Normality and finite automata

Definition

A sequence $x = a_1 a_2 a_3 \cdots$ is *compressible* by a transducer if and only if its accepting run $q_0 \xrightarrow{a_1 | v_1} q_1 \xrightarrow{a_2 | v_2} q_2 \xrightarrow{a_3 | v_3} q_3 \cdots$ satisfies

$$\liminf_{n \to \infty} \frac{|v_1 v_2 \cdots v_n|}{n} \frac{\log |B|}{\log |A|} < 1.$$

Normality and finite automata

Let $L \subseteq A^*$. The infinite word obtained by *prefix-selection* by L is $a_{p(1)}a_{p(2)}\cdots$, where $p(j)$ is the j-th in sorted $\{i \ : \ a_1a_2\cdots a_{i-1} \in L\}$.

Let $X \subseteq A^\omega$. The infinite word obtained by *suffix-selection* by X is $a_{s(1)}a_{s(2)}\cdots$, where $s(j)$ is the j-th in sorted $\{i : a_{i+1}a_{i+2}\cdots \in X\}$.

Borel hierarchy for subsets of the real numbers

Recall Borel hierarchy for subsets of the real numbers is the stratification of the σ-algebra generated by the open sets with the usual interval topology.

- A set is $\mathbf{\Sigma}^0_1$ iff it is open.
- A set is $\mathbf{\Pi}^0_1$ iff it is closed.
- A set is $\mathbf{\Sigma}^0_{n+1}$ iff it is countable union of $\mathbf{\Pi}^0_n$ sets.
- A set is $\mathbf{\Pi}^0_{n+1}$ iff it is a countable intersection of $\mathbf{\Sigma}^0_n$ sets.

A set is is hard for a Borel class if every set in the class is reducible to it by a continuous map.

A set is complete for a class if it is hard for this class and belongs to the class.

Effective Borel hierarchy

Arithmetic hierarchy of formulas in the language of second-order arithmetic:

- formulas involve only quantification over integers.
- atomic formulas assert algebraic identities between integers or membership of real numbers in intervals with rational endpoints.
- a formula is Π^0_0 and Σ^0_0 if all its quantifiers are bounded.
- a formula is Σ^0_{n+1} if it has the form $\exists x\, \theta$ where θ is Π^0_n.
- a formula is Π^0_{n+1} if it has the form $\forall x\, \theta$ where θ is Σ^0_n.

A set of real numbers is Σ^0_n (respectively Π^0_n) in the effective Borel hierarchy if membership in that set is definable by a formula Σ^0_n (respectively Π^0_n).

Effective reductions are computable maps.

Normality in the effective Borel hierarchy

A real number x is normal to base b if, and only if,

- for every $k \geq 1$, x is simply normal to base b^k,
- for every $k \geq 1$, and for every block d of k digits,

$$\lim_{n\to\infty} \frac{\#occ(a_1 a_2 \ldots a_n, d)}{n} = \frac{1}{b^k}$$

where $a_1 a_2 a_3 \ldots$ is the expansion of x in base b. Thus,

$$\forall k \; \forall d \; \forall \epsilon \; \exists n \; \forall m \geq n \;\; \theta(a_1 a_2 .. a_m, b)$$

where $\theta(a_1 a_2 .. a_m, b)$ is computable in x.

This is $\forall\exists\forall$ formula with one free real x and one free integer b, quantification only on the integers.

Normality in the effective Borel Hierarchy

In the effective Borel hierarchy for susbets of real numbers,

- Normal to a fixed base b is Π^0_3.
- Normal to all bases is also Π^0_3.
- Normal to some base is Σ^0_4.

Normal to all bases is complete at the third level

Asked first by Kechris 1994.

Theorem (Ki and Linton 1994)

The set of real numbers that are normal to any fixed base is Π^0_3-complete.

Theorem (Becher, Heiber, Slaman 2014)

The set of real numbers that are absolutely normal is Π^0_3-complete.

Effective case implies the general case

- Every Σ^0_n set is $\mathbf{\Sigma}^0_n$ and every Π^0_n set is $\mathbf{\Pi}^0_n$.
- For every $\mathbf{\Sigma}^0_n$ set A there is a Σ^0_n formula and a real parameter such that membership in A is defined by that Σ^0_n formula relative to that real parameter.

Since computable maps are continuous, proofs of hardness in the effective hierarchy yield proofs of hardness in general by relativization.