Perfect necklaces and their discrepancy

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Discrepancy in discrete spaces

Flajolet, Kirschenhofer and Tichy 1988.

Let A a finite alphabet with uniform measure.
Let $\ell : \mathbb{N} \to \mathbb{N}$ be non decreasing but $\ell(N) < N$.

An infinite sequence x is $\ell(N)$-u.d. if $\Delta_{\ell(N),N}(x) = o(|A|^{-\ell(N)})$,

where

$$\Delta_{\ell,N}(a_1 a_2 \ldots) = \max_{u \in A^\ell} \left| \frac{|a_1 a_2 \ldots a_{N+\ell-1}|_u}{N} - \frac{1}{|A|^\ell} \right|$$

On u.d. sequences

Theorem (Flajolet, Kirschenhofer and Tichy 1988)

Let alphabet A with uniform measure. For every positive ε almost all infinite sequences of symbols in A, w.r.t. product measure are $\ell(N)$-u.d. for $\ell(N) = \lfloor (1-\varepsilon) \log N \rfloor$.

Problem

What is the minimum discrepancy $\Delta_{\ell(N),N}(x)$ for some sequence x?

This is the discrete counterpart of Korobov's question on the minimum $D_N(\{b^n x\})_{n \geq 0}$ for some real number x and integer base b.

Construction of u.d.sequences

Construction by concatenating of perfect necklaces of increasing size.

Study families of perfect necklaces and their discrepancy.

Count the number of elements in the family.

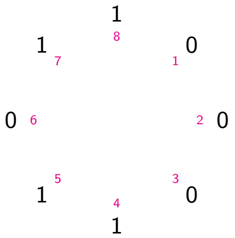
Observation

All blocks of length n concatenated in lexicographical order, view it circularly.

For example, for alphabet $\{0, 1\}$.

$n = 2$

$00\ 01\ 10\ 11$



00 occurs twice (p:1,2);

01 occurs twice (p:3,6);

10 occurs twice (p:5,8);

11 occurs twice (p:4,7)

Each block of length n occurs n times at positions with different residue modulo n.

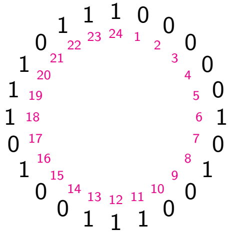
Observation

All blocks of length n concatenated in lexicographical order, view it circularly.

For example, for alphabet $\{0, 1\}$.

$n = 3$

$000\ 001\ 010\ 011\ 100\ 101\ 110\ 111$



000 occurs three times (p:1,2,3)

001 occurs three times (p:4,9,14)

...

Each block of length n occurs n times at positions with different residue modulo n.

Observation

Not every permutation of the blocks of length n has the property:

00 10 11 01

000 101 001 010 011 100 110 111

Perfect necklaces

A necklace is the equivalence class of a word under rotations.

Definition (Alvarez, Becher, Ferrari and Yuhjtman 2016)

A necklace over a b-symbol alphabet is (n,k)-perfect if each block of length n occurs k times, at positions with different modulo k, for any convention of the starting point.

The (n,k)-perfect necklaces have length kb^n.

De Bruijn circular sequences are exactly the $(n,1)$-perfect necklaces.

The ordered necklace

The concatenation blocks of length n over a b-symbol alphabet in lexicographic order is called the ordered necklace for length n.

$$00\ 01\ 10\ 11$$

Proposition (Alvarez, Becher, Ferrari and Yuhjtman 2016)

The ordered necklace for length n is (n, n)-perfect.

Observation

For the ordered (n, n)-perfect necklace $\Delta_{n,N}(x)$ is exactly order $1/\log N$.
It follows from Schiffer's proof taking $f(n) = n$ and $d_n = 0$.

Theorem (Schiffer 1986)

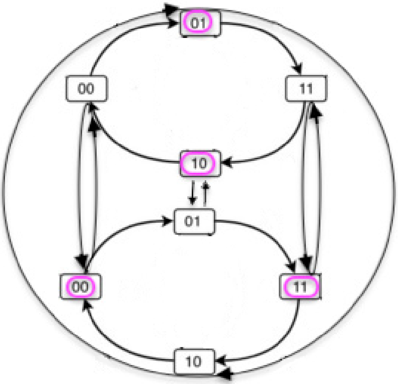
Let $f(n)$ be polynomial in n with rational coefficients, not constant, and let $(d_n)_{n\geq 1}$ be a bounded of rational numbers such that $f(n) + d_n$ is positive integer. For for $x = f(1 + d_1)f(2 + d_2)\ldots, \Delta_N(x)$ is exactly in the order of $1/\log N$.

Astute graphs

Fix b-symbol alphabet.
The astute graph $G_{b,n,k} = (V, E)$ is directed, with kb^n vertices.

$V = \{0, .., b-1\}^n \times \{0, .., k-1\}$.
$E = \{(w, m) \to (w', m') : w[2, n] = w'[1, n-1]$ and $(m+1) \mod k = m'\}$

$G_{2,2,2}$



Astute graphs

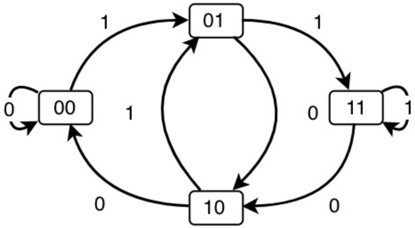
Observation

Every (n, k)-perfect necklace corresponds to some Hamitonian cycle in $G_{b,n,k}$.

Observation

$G_{b,n,1}$ *is the de Bruijn graph of blocks of length n over b-symbols.*

$G_{2,2,1}$



Perfect necklaces in astute graphs

Every Hamiltonian cycle in $G_{b,n,k}$ is Eulerian in $G_{b,n-1,k}$,
because $G_{b,n,k}$ is the line graph of $G_{b,n-1,k}$.

Each Eulerian cycle in $G_{b,n-1,k}$ gives one (n,k)-perfect necklace.

Each (n,k)-perfect necklace can come from many Eulerian cycles in $G_{b,n-1,k}$

Theorem (Alvarez, Becher, Ferrari and Yuhjtman 2016)

The number of (n,k)-perfect necklaces over a b-symbol alphabet is

$$\frac{1}{k}\sum_{d_{b,k}|j|k} e(j)\varphi(k/j)$$

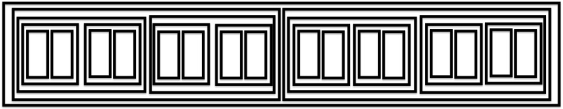
where

- $d_{b,k} = \prod p_i^{\alpha_i}$, *where p_i divides both b and k, and α_i is the exponent of p_i in the factorization of k,*
- $e(j) = (b!)^{jb^{n-1}} b^{-n}$ *is the number of Eulerian cycles in $G_{b,n-1,j}$*
- φ *is Euler's totient function*

Nested perfect necklaces

Definition

An (n, k)-perfect necklace over a b-symbol alphabet is nested if $n = 1$ or it is the concatenation of b nested $(n-1, k)$-perfect necklaces.



This is a nested $(2, 2)$-perfect necklace for $b = 2$

$$\underbrace{0011}_{\text{(1,2)-perfect}}\ \underbrace{0110}_{\text{(1,2)-perfect}}$$

The ordered perfect necklace is not nested.
For example for $b = 3$ and $n = 2$,

$$\underbrace{00\ 01\ 02}_{\text{not (1,2)-perfect}}\ \underbrace{10\ 11\ 12}_{\text{not (1,2)-perfect}}\ \underbrace{20\ 21\ 22}_{\text{not (1,2)-perfect}}$$

Nested perfect necklaces

For example, in the binary alphabet and n is a power of 2,

1	nested (n, n)-perfect necklace	determines
2	nested $(n-1, n)$-perfect necklaces	determine
2^2	nested $(n-2, n)$-perfect necklaces	determine
...		
2^{n-1}	$(1, n)$-perfect necklaces	

Nested perfect necklaces exist

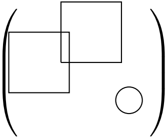
For n a power of 2, M.Levin (1999) defines a matrix M in $\mathbb{F}_2^{n\times n}$ using Pascal triangle matrix modulo 2,

$$M := (p_{i,j})_{i,j=0,1,\ldots n-1} \text{ where } p_{i,j} := \binom{i+j}{j} \mod 2.$$

M is upper triangular and it has the following property on submatrices.

Lemma (Levin 1999 from Bicknell and Hoggart 1978; Mereb 2022)

For Pascal triangle matrix modulo 2, each square submatrix at the left or at the top has determinant computed in $\mathbb{Z}$ equal to 1 or -1.

$M =$ 

Then, if these determinants are computed in $\mathbb{Z}/b\mathbb{Z}$, for any integer $b \geq 2$, they are equal to 1 or -1.

Nested (n, n)-perfect necklaces exist

Definition (Levin's definition 1999)

Let integer $b \geq 2$ and let n be a power of 2.

Identify the set of non-negative integers modulo b^n according to representation in base b with the vectors $w_0, \ldots w_{b^n-1}$ in $(\mathbb{Z}/b\mathbb{Z})^n$.

Let $M \in \mathbb{F}_2^{n\times n}$ be the Pascal matrix modulo 2.
Define the necklace $Mw_0 \ldots Mw_{b^n-1}$, computation is done in $\mathbb{Z}/b\mathbb{Z}$.

For example, for $b = 2$,

$n = 2^0$ 0 1
$n = 2^1$ 00 11 10 01
$n = 2^2$ 0000 1111 1010 0101 1100 0011 0110 1001 1000 0111 0010 1101 0100 1011 1110
...

Theorem (Becher and Carton 2019)

Let $b \geq 2$ be a integer and let n be a power of 2. The necklace given by the Pascal matrix modulo 2 is nested (n, n)-perfect.

How many nested (n, n)-perfect necklaces, n power of 2?

Theorem (Becher and Carton 2019)

There are 2^{2n-1} binary nested (n, n)-perfect necklaces, n power of 2.

Problem

What is the number of (n, n)-perfect necklaces for $b \geq 3$ and n a power of 2?

The number of binary nested perfect necklaces, n power of 2

Definition (Pascal-like family $\mathcal{P} \subseteq \mathbb{F}_2^{n\times n}$)

Let n be a power of 2.
Let $(\eta_j)_{0\leq j<n}$ such that $\eta_0 = 0, \eta_j \leq \eta_{j+1} \leq \eta_j + 1$.
Define $M^\eta = (p^\eta_{i,j})_{0\leq i,j<n}$ in $\mathbb{F}_2^{n\times n}$,

$$p^\eta_{i,j} = \binom{i+j-\eta_j}{j} \mod 2$$

For each M in $\mathcal{P}$ and for each vector z in $\mathbb{F}_2^n$, we have
$M(w_0 \oplus z) \dots M(w_{2^n-1} \oplus z)$ is a binary nested (n, n)-perfect necklace.
If $z' = Mz$, $\quad M(w_0 \oplus z) \dots M(w_{2^n-1} \oplus z) = Mw_0 \oplus z' \dots Mw_{2^n-1} \oplus z'$,

Thus, we count

$\#$ matrices $\in \mathbb{F}_2^{n\times n}$ in $\mathcal{P}\times$ $\#$ vectors $z \in \mathbb{F}_2^n = 2^{2n-1} \leq$ Total.

By a graph theoretical argument we know that there can be no more.

Discrepancy of nested perfect necklaces, n power of 2

Theorem (Levin 1999; Becher and Carton 2019)

Let $b \geq 2$ be an integer and let n be a power of 2.
Each nested (n, n)-perfect necklace given by Pascal-like matrix modulo 2,
$\Delta_N = O((n \log N)/N)$.

Discrepancy of nested perfect necklaces, n power of 2

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Theorem (Hofer and Larcher 2022)

Let $b = 2$ and let n be a power of 2.
Each nested (n, n)-perfect necklace given by Pascal-like matrix modulo 2,
Δ_N *is exactly of order* $n(\log N)/N$*.*

Discrepancy of nested perfect necklaces, n power of 2

Theorem (Levin 1999; Becher and Carton 2019)

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Corollary (Hofer and Larcher 2022)

Levin's real number x whose binary expansion is the concatenation of nested (n,n)-perfect necklaces for n increasing powers of 2,
$D_N(\{2^t x\}_{t\geq 0})$ *is exactly of order* $(\log N)^2/N$.

Discrepancy of nested perfect necklaces, n power of 2

Theorem (Levin 1999; Becher and Carton 2019)

Let $b \geq 2$ be an integer and let n be a power of 2.
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Problem (work in progress)

What is the exact order of discrepancy for the nested (n, n)-perfect necklaces, n power of 2, for $b \geq 3$ given by Pascal matrix modulo 2?

Nested (n,n)-perfect necklaces, prime base b, n prime power

Definition (Pascal-like family $\mathcal{P} \subseteq \mathbb{F}_b^{n\times n}$ Hofer and Larcher, 2022)

Let b be a prime.
Let n be a power of b.
Let $(u_j)_{0\le j<n}$ with $u_j \not\equiv 0 \mod b$.
Let $(\eta_j)_{0\le j<n}$ such that $\eta_0 = 0, \eta_j \le \eta_{j+1} \le \eta_j + 1$.
Define $M^{u,\eta} = (p^{u,\eta}_{i,j})_{0\le i,j<n}$ in $\mathbb{F}_b^{n\times n}$,

$$p^{u,\eta}_{i,j} = \binom{i+j-\eta_j}{j} u_j \mod b.$$

Exact order of discrepancy for prime bases

Theorem (Hofer and Larcher 2022)

Let b be a prime. For every real number x whose base-b expansion is the concatenation of nested (n,n)-perfect necklaces given by Pascal-like matrices modulo b, for n increasing powers of b, $D_N(\{b^t x\}_{t\geq 0})$ is exactly in the order of $(\log N)^2/N$.

Problem

How many nested (n,n)-perfect necklaces in base b, b prime, n a power of b? What is their discrepancy?

Arithmetic perfect necklaces

Identify the blocks of length n over a b-symbol alphabet with the set of non-negative integers modulo b^n according to representation in base b.

Definition

Let $b \geq 2$ be an integer, let r be coprime with b. Let n be a positive integer. An arithmetic necklace is the concatenation of blocks of length n corresponding to

$$0, r, 2r, \ldots, (b^n - 1)r \qquad \bmod b^n$$

With $r = 1$ we obtain for each n the lexicographically ordered necklace:

$$00\ 01\ 10\ 11; \qquad 00\ 01\ 02\ 10\ 11\ 12\ 20\ 21\ 22$$

Theorem (Alvarez, Becher, Ferrari and Yuhjtman 2016)

For each n, arithmetic necklaces are (n, n)-perfect.

On the discrepancy of arithmetic necklaces

Observation

In general, Schiffer's $(f(1))(f(2))\ldots,(f(b^n-1))$ *is not* (n,n)*-perfect. because for arbitrary arithmetic progressions, the representation of* $f(n)$ *requires more than* n *symbols.*

Theorem (Schiffer 1986)

Let $f(n)$ *be polynomial in* n *with rational coefficients, not constant, and let* $(d_n)_{n\geq 1}$ *be a bounded of rational numbers such that* $f(n)+d_n$ *is positive integer. For for* $x = f(1+d_1)f(2+d_2)\ldots,\Delta_N(x)$ *is exactly in the order of* $1/\log N$.

Classical discrepancy on arithmetic progressions

For $\alpha \in \mathbb{R}$, let $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ be the fractional part of α.
Let $\alpha = [a_0; a_1, a_2, \ldots]$ irrational. For every positive integer N,

$$N D_N(\{n\alpha\}_{n \geq 1}) \leq \sum_{j=1}^{\ell(N)+1} a_j.$$

where $q_{\ell(N)} \leq N \leq q_{\ell(N)+1}$.

Levin's arithmetic progression with small discrepancy

If $\alpha = [a_0; a_1, \ldots, a_n]$ we write $S(\alpha) = \sum_{i=1}^{n} a_i$.

Theorem (Levin 1999 Theorem 1, using Popov 1981)

Let $b \geq 2$. For every m there is r coprime with b such that

$$\sum_{i=1}^{m} S(r/b^i) < Km^3, \text{ where } K \text{ is constant.}$$

Thus, for such a (m,m)-arithmetic necklace, Δ_N is $O(m^3)$.

Levin's construction (Theorem 1, 1999) is the concatenation of (n,n)-perfect arithmetic necklaces, $n = 1, 2, 3$.

Levin proves if N is at the (m,m)-necklace,

$$ND_N(\{b^n x\}_{n\geq 1}) < 4(\sum_{i=1}^{m} ND_N(\{nr/b^i\}_{n\geq 0}) + m) = O((\log N)^3)$$

Conjecture

Let $b \geq 2$. For every m there is r coprime with b such that

$$\sum_{i=1}^{m} S(r/b^i) < Km^2 \log m, \text{ where } K \text{ is constant.}$$

These arthmetic necklaces are (m, m)-perfect have $\Delta_N = O(m^2 \log m)$. Hence, for x whose base b-expansion is the concatenation of these arithmetic necklaces of increasing order,

$$ND_N(\{b^n x\}_{n \geq 1}) = O((\log N)^2 \log \log N).$$

Significant related results

Since $S(A/B) \simeq S(B/A)$, we need:

For every m,
find r coprime with b, r between 1 and $b^m - 1$
that minimizes $S(b/r) + S(b^2/r) + \ldots + S(b^m/r)$.

Let $M(x)$ the maximum coefficient in the finite continued fraction for x.

Theorem (Aistleitner, Borda, Hauke 2022 Corollary 3 and 4)

For all integers $r \geq 3$ there are reduced fractions a/r and c/r such that

$$S(a/r) \leq \frac{12}{\pi^2} \log r \ \log\log r + O(\log r),$$

$$M(c/r) \leq \frac{12}{\pi^2} \log r + O((\log\log r)^3).$$

This improves Larcher 1986; Chang 2011; Rukavishnikova 2011.

Significant related results

For the case of $b = 2$.

For every m, find r odd between 1 and $2^m - 1$ that minimizes $S(r/2) + S(r/2^2) + \ldots + S(r/2^m)$

Theorem (Neiderreter 1986 –Zaremba conjecture for the powers of 2–)

For all N power of 2 there is a reduced fraction a/N such that $M(a/N) < 3$.

There is a minimizer

Definition (minimizer)

Let $b \geq 2$ be an integer and let m be a positive integer.

A minimizer for (b, m) is a positive integer r that minimizes $\sum_{i=1}^{m} S(r/b^i)$.

	$b = 2$		$b = 3$		$b = 10$	
m	r	$\sum_{i=1}^{m} S(r/b^i)$	r	$\sum_{i=1}^{m} S(r/b^i)$	r	$\sum_{i=1}^{m} S(r/b^i)$
1	1	2	1	3	3	6
2	1	6	2	9	27	17
3	3	11	5	18	173	36
4	3	19	31	29	2627	62
5	5	29	92	44	22627	91
6	19	39	140	63	262113	128
7	37	52	857	85	2262113	170
8	45	67	2570	109	16172177	227
9	151	83	9131	138	226542279	286
10	151	102	12262	172	–	–
11	807	125	31907	207	–	–
12	867	151	46787	245	–	–
13	3367	174	311411	286	–	–
14	3433	201	1288610	332	–	–
15	4825	231	3761986	379	–	–
16	13893	260	–	–	–	–
17	51351	289	–	–	–	–

Usually four distinct minimizers

Observation

Let b and r_1 be coprime positive integers such that $b \geq 2$ and let m be an integer such that $1 \leq r_1 < b^m$. Then, the numbers

$$r_2 = b^m - r_1$$
$$r_3 \text{ such that } r_3 r_1 \equiv 1 \pmod{b^m} \text{ and } 1 \leq r_3 < b^m$$
$$r_4 = b^m - r_3$$

satisfy $\displaystyle\sum_{i=1}^{m} S(r_1/b^i) = \sum_{i=1}^{m} S(r_j/b^i)$ *for* $j = 2, 3, 4.$

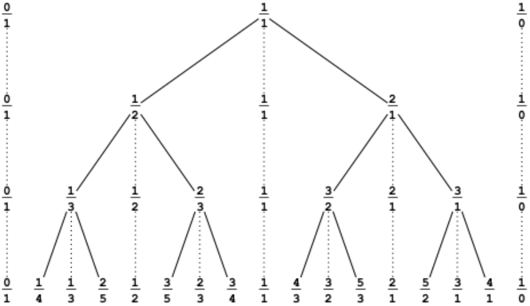
Thus, if r_1 is a minimizer for (b, m) then r_2, r_3, r_4 are so too.

Because mirror continued fractions have the same S, and
the continued fractions of x and $1 - x$ have the same S.

Let $x = [0; a_1, a_2, \ldots]$. $1 - x = [0; 1, a_1 - 1, a_2,]$ if $a_1 > 1$; $[0; a_2 + 1, a_3,]$ if $a_1 = 1$.

Stern-Brocot tree

The Stern-Brocot tree is a binary tree whose vertices are the positive rational numbers augmented $0/1$ and $1/0$. The root is 1 (row $d = 0$). The left subtree, the Farey tree, contains the rationals less than 1.



The largest denominator at row d is Fibonacci$(d+1)$.
The number x is at row d if and only if $S(x) = d + 1$.
For b and m, find r coprime with b, between 1 and $b^m - 1$ minimizing

$$\sum_{i=1}^{m} \text{row}(r/b^i).$$

List of minimizers for base $b = 2$

m	r_1	r_2	r_3	r_4	has more
1	1	1	1	1	
2	1	3	1	3	
3	3	5	3	5	
4	3	13	11	5	
5	5	27	13	19	
6	19	45	27	37	
7	37	91	45	83	
8	45	211	165	91	
9	151	361	295	217	
10	151	873	807	217	
11	807	1241	1175	873	
12	867	3229	1611	2485	
13	3367	4825	4759	3433	
14	3433	12951	4825	11559	
15	4825	27943	19817	12951	True
16	13893	51643	27789	37747	True
17	51351	79721	79655	51417	
18	79655	182489	182423	79721	
19	79655	444633	444567	79721	
20	444567	604009	603943	444633	
21	444567	1652585	1652519	444633	
22	444567	3749737	3749671	444633	

List of minimizers for base $b = 3$

m	r_1	r_2	r_3	r_4	has more
1	1	2	1	2	
2	2	7	5	4	
3	5	22	11	16	
4	31	50	34	47	
5	92	151	140	103	
6	140	589	578	151	True
7	857	1330	860	1327	
8	2570	3991	3980	2581	
9	9131	10552	10541	9142	
10	12262	46787	42883	16166	
11	31907	145240	68306	108841	
12	46787	484654	311411	220030	
13	311411	1282912	1109669	484654	True
14	1288610	3494359	1719647	3063322	
15	3761986	10586921	3794110	10554797	
16	7547866	35498855	30542071	12504650	
17	30041471	99098692	94985393	34154770	True

List of minimizers for base $b = 10$

m	r_1	r_2	r_3	r_4	has more
1	3	7	7	3	
2	27	73	63	37	
3	173	827	237	763	True
4	2627	7373	3563	6437	True
5	22627	77373	43563	56437	
6	262113	737887	262177	737823	True
7	2262113	7737887	2262177	7737823	
8	16172177	83827823	65472113	34527887	True
9	226542279	773457721	742147319	257852681	

Simple perfect necklaces

Definition

The simple (n, k)-perfect necklaces correspond to Eulerian cycles in $G_{b,n-1,k}$ that are defined by joining disjoint simple cycles.

We extend the shift registers of Golomb 1967 to construct some: (n, k)-rotation cycles and (n, k)-increment cycles.

Observation

For every n, the ordered necklace of blocks of length n is an arithmetic and simple (n, n)-perfect necklace.

Classical Lyndon words are the lexicographically least representatative in equivalence classes of word rotations. Their concatenation yields the lexicografically greatest de Bruijn sequence (Fredericksen algorithm). Define Lyndon words for pairs, adapt Fredericksen's algorithm.

Simple perfect necklaces

Theorem (Álvarez, Becher, Mereb, Pajor and Soto 2022)

We construct an (n, n)*-perfect necklace by joining* (n, k)*-increment cycles such that,*

for $b = 2$*,* $\Delta_{1,N} \leq (n + 1)/2$*;*
for $b \geq 3$*,* $\Delta_{1,N} \leq 2n\dfrac{b-1}{b}$*,*
and there is N *such that* $\Delta_{1,N} \geq \dfrac{n}{2}\dfrac{b-1}{b}$*.*

Summary on perfect necklaces and discrepancy

De Bruijn $(n,1)$-perfect necklaces
$\Delta_N = O(\sqrt{(\log\log N)/(\log N)})$, Flajolet, Kirschenhofer, Tichy 1989; Ugalde 2000

Nested (n,n)-perfect necklaces,
n a power of 2, $b \geq 2$, $\Delta_N = O((n \log N)/N)$, Levin 1999; Becher and Carton 2019
n a power of b, b prime, family, Δ_N exact order $(n \log N)/N$ Hofer and Larcher 2022

Arithmetic (n,n)-perfect necklaces
Ordered necklace Δ_N is exactly of order $1/\log N$, Schiffer 1986
Exist arithmetic (n,n)-perfect $\Delta_N = O(n^3/N)$, Levin 1999
Conjecture: Exist arithmetic (n,n)-perfect $\Delta_N = O((n^2 \log n)/N)$

Simple (n,n)-perfect necklaces Alvarez,Becher,Mereb,Pajor,Soto 2022
There exist simple (n,n)-perfect with $\Delta_{1,N}$ is exactly of order n.

There are many open questions.

Some of the references

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Classical definitions

Let b be a base. Let $\Sigma_b = \{0, \ldots b-1\}$.

Rotation $r : \Sigma_b^n \to \Sigma_b^n$ moves the last character of the string to the front.
For example: $r(0010) = 0001; r(0001) = 1000; r^2(0010) = 1000$
The equivalence classes of Σ_b^n under rotation are the necklaces of size n.

Since r is invertible with $r^{-1} = r^{n-1}$, $\langle r \rangle$ is a group that acts on Σ_b^n.

Incremented rotation $\iota : \Sigma_b^n \to \Sigma_b^n$ increments the last character of the string (modulo b) and moves that incremented character to the front.
For example, if $b = 3$, we have: $i(0021) = 2002$, $i(2002) = 0200$

Since ι is invertible, $\langle \iota \rangle$ is a group that acts on Σ_b^n.

Simple perfect necklaces

Definition

For positive n and k we define $r_k : \Sigma_b^n \times \mathbb{Z}/k\mathbb{Z} \to \Sigma_b^n \times \mathbb{Z}/k\mathbb{Z}$,

$$r_k(s,t) = (r(s), t+1)$$

The (n,k)-rotation necklaces are the orbits of $\langle r_k \rangle$ on the set $\Sigma_b^n \times \mathbb{Z}/k\mathbb{Z}$

Applying Burnside's Lemma,

Proposition (ABMPS 2022)

The (n,k)-rotation necklaces correspond to some simple cycles in graph $G_{b,n-1,k}$ and determine a partition of $G_{b,n-1,k}$. The total number is

$$\frac{\gcd(n,k)}{n} \sum_{\gcd(n,k)|d|n} \varphi(n/d) b^d$$

Simple perfect necklaces

Definition

For a positive integer k we define $\iota_k : \Sigma_b^n \times \mathbb{Z}/k\mathbb{Z} \to \Sigma_b^n \times \mathbb{Z}/k\mathbb{Z}$,

$$\iota_k(s,t) = (\iota(s), t+1)$$

The (n,k)-increment necklaces are the orbits of $\langle \iota_k \rangle$ on the set $\Sigma_b^n \times \mathbb{Z}/k\mathbb{Z}$
Applying Burnside's Lemma,

Proposition (ABMPS 2022)

The (n,k)-increment necklaces correspond to some simple cycles in graph $G_{b,n-1,k}$ and determine a partition of $G_{b,n-1,k}$.
The total number is

$$\frac{\gcd(\gcd(k,bs)n/s, kb)}{nb} \sum_{\gcd(n,\mathrm{lcm}(k,bs))|d|n} \varphi(n/d) \cdot b^d$$

where s is the smallest divisor of n such that n/s is coprime with b.

Nested marvelous (semi-perfect) necklaces

Definition

A necklace over a a b-symbol alphabet is nested (n, k)-marvelous if all blocks of length n occur exactly k times, and in case $n > 1$ it is the concatenation of b nested $(n-1, k)$-marvelous necklaces.

This is nested $(3, 3)$-marvelous, not perfect,

$$000111\ 011001\ 000111\ 101010$$

Theorem (Frizzo 2020; Larcher and Hofer 2022)

For every number x whose base-b expansion is the concatenation of nested (n, n)-marvelous necklaces, for n a power of b or a power of 2, $D_N(\{b^t x\}_{t\geq 0})$ is $O\left((\log N)^2/N\right)$.