Normality, Computability and Diophantine Approximation

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Irrationality exponent

Definition (originating with Liouville 1855)

The irrationality exponent of a real number x, is the supremum of the set of real numbers z for which

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

Irrationality exponent

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- Rational numbers have irrationality exponent equal to 1.

Computable real numbers

A real x is computable if there is a computable sequence of rationals $(r_j)_{j \geq 0}$ such that for all j, $|x - r_j| < 2^{-j}$.

Irrationality exponents and computable numbers

Theorem (Bugeaud 2008)

Let a and r be reals greater than or equal to 2. The number

$$x(a,r) = \sum_{j \geq 1} \frac{2}{3^{\lfloor r \ a^j \rfloor}}$$

has irrationality exponent equal to a.

This proves that the middle third Cantor set contains uncountably many reals whose irrationality exponent is equal to a, each computable from a and r.

Irrationality exponents and computable numbers

Bugeaud asked:
Are there computable numbers with non-computable irrationality exponents?

First announcement

Characterization of the irrationality exponents of computable numbers.

Theorem (Becher, Bugeaud, Slaman 2014)

Let a be a real greater than or equal to 2. Then, the real a is the irrationality exponent of some computable number if and only if a is right computably enumerable from $0'$.

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A real a is right-computably enumerable from $0'$ if and only if there is a computable doubly-indexed sequence $a(j, s)$ of rational numbers such that

$$(\lim_{s\to\infty} a(j,s))_{j\geq 0} \text{ is strictly decreasing and } \lim_{j\to\infty} \lim_{s\to\infty} a(j,s) = a.$$

Jarník - Besicovich Theorem

Theorem (Jarník 1929, Besicovich 1934)

For every real a greater than or equal to 2, the set of reals with irrationality exponent equal to a has Hausdorff dimension $2/a$.

Jarník's fractal

Fix a real a greater than 2. Jarník gave a Cantor-like construction of a set in $[0,1]$. Let $(m_k)_{k\geq 1}$ be an appropriate increasing sequence of positive integers.

For each $k \geq 1$,

$$E(k) = \dot{\bigcup_{\substack{q \text{ prime} \\ m_k < q < 2m_k}}} \left\{ x \in \left(\frac{1}{q^a}, 1 - \frac{1}{q^a} \right) : \exists p \in \mathbb{N}, \left| \frac{p}{q} - x \right| < \frac{1}{q^a} \right\}$$

$E(k)$ has about $\frac{m_k^2}{\log m_k}$ disjoint intervals, each of length at least $\frac{2}{(2m_k)^a}$.

Not all the intervals at level k have the same length (the smaller the denominator the larger the interval).

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Jarník's fractal for the real a is $\qquad J = \bigcap_{k \geq 1} E(k).$

Jarník's fractal

For each real a greater than 2, Jarník's fractal J in $[0, 1]$ has Hausdorff dimension $2/a$ and the uniform measure ν on J satisfies:

- the set of reals with irrationality exponent equal to a has ν-measure equal to 1.
- for each real b greater than a, the set of reals with irrationality exponent equal to b has ν-measure equal to 0.

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Theorem (Mass Distribution Principle)

Let ν be a finite measure, d a positive real number. Suppose that there are positive reals c and ϵ such that for all intervals I with $|I| < \epsilon$,

$$\nu(I) \leq c\,|I|^d.$$

Then, for any set X with Hausdorff dimension less than d, we have $\nu(X) = 0$.

Take $d = 2/a$.

Normal numbers

A *base* is an integer b greater than or equal to 2.

Definition (Borel, 1909)

A real x is simply normal to base b if in the expansion of x in base b, each digit occurs with limiting frequency equal to $1/b$.

A real x is normal to base b if x is simply normal to every base b^j, for every j.

A real x is absolutely normal if x is normal to every base.

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Any number in the Cantor middle third is not simply normal to base 3,
for instance, $x(a,r) = \sum_{j\geq 1} \frac{2}{3^{\lfloor r\ a^j \rfloor}}$, for $a \geq 2$ and $r \geq 2$.

Existence of absolutely normal numbers

Theorem (Borel 1909)

The set of absolutely normal numbers has Lebesgue measure one.

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Constructions of absolutely normal numbers (1917-2013) accounted for no other mathematical (geometric, algebraic, number-theoretic) properties.

Computing absolutely normal numbers

Theorem (Lutz, Mayordomo 2013; Figueira, Nies 2013; Becher, Heiber, Slaman 2013)

There is a polynomial-time algorithm to compute an absolutely normal number.

Normality as uniform distribution modulo one

Theorem (Wall 1949)

A real x is normal to base b if, and only if, $(b^k x)_{k\geq 0}$ equidistributes modulo 1 for Lebesgue measure.

Belief

Typical elements of well-structured sets, with respect to appropriate measures, are absolutely normal, unless the set displays an obvious obstruction.

Taking this to the extreme and applying it to singletons one arrives at the folklore conjecture that constants such as π, e, $\sqrt{2}$ are absolutely normal.

Existence of absolutely normal Liouville numbers

- (Kaufman 1981) For any real a greater than 2, there is a measure ν on the Jarník fractal for a such that the Fourier transform of ν decays quickly.
- (Bluhm 2000) There is a measure ν supported by the Liouville numbers such that the Fourier transform of ν decays quickly.

Theorem (Bugeaud 2002)

There is an absolutely normal Liouville number.

Computing absolutely normal Liouville numbers

Theorem (Becher, Heiber, Slaman 2013)

There is a computable absolutely normal Liouville number.

Second announcement

Computing absolutely normal numbers with prescribed irrationality exponent.

Theorem (Becher, Bugeaud, Slaman 2014)

For every real a greater than or equal to 2 there is a real x such that

- x *has irrationality exponent equal to the given real* a,
- x *is absolutely normal,*
- x *is computable from* a.

Computing numbers that are normal to different bases

Two integers are multiplicatively dependent if one is a rational power of the other. Example: 2 and 8 are multiplicatively dependent. But 2 and 6 are not.

Theorem (Cassels 1959; Schmidt 1961/1962; Becher, Slaman 2013)

For any subset of the multiplicative dependence classes, there is a real x which is normal to the bases in the given subset, and not simply normal to the bases in its complement.

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Theorem (Becher, Bugeaud, Slaman 2013)

For any function M from the multiplicative dependence classes to subsets of positive integers such that

- *for each b, if $b^{km} \in M(b)$ then $b^m \in M(b)$.*
- *if $M(b)$ is infinite then $M(b) = \{b^m : m \geq 1\}$.*

there is a real x such that x is simply normal to exactly the bases in the subsets.

Second announcement, in full

Computing numbers with prescribed irrationality exponent and pattern of normality.

Theorem (Becher, Bugeaud, Slaman 2014)

For every real a greater than or equal to 2 and every set of bases satisfying the conditions for simple normality, there is a real number x with irrationality exponent a and simply normal to exactly the bases in the specified set.

Theodore Slaman says

CCR and Diophantine Approximation are natural partners. I hope everyone will consider this talk to be an invitation to explore the connections.

To be continued ...

Open Questions

Automatic numbers and irrationality exponents

Question (Bugeaud)

Is the irrationality exponent of an automatic real number always rational?

Recall that an automatic real number is a real whose expansion in some base can be generated by a finite automaton.

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Is the irrationality exponent of an automatic real number always rational?

Recall that an automatic real number is a real whose expansion in some base can be generated by a finite automaton.

- There are automatic real numbers with any prescribed rational irrationality exponent (proved by Bugeaud).
- The irrationality exponent of every Thue- Morse- Mahler number is equal to 2 (proved by Bugeaud). For instance, Morse-Thue constant in base 2 is defined by $a_{n+1} = a_n + \overline{a_n}\ 2^{-2^n}$.
- The irrationality exponent of an automatic number is finite (proved by Adamczewski and Cassaigne).

Normality, non-normality and irrationality exponents

Question (Bugeaud)

Does the middle third Cantor set contain a Liouville number which is normal to base 2?

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Theorem (Cassels, 1959; Schmidt 1960)

Almost every real number in the middle third Cantor set is normal to every base which is not a power of 3.

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