Partial characterizations of clique-perfect graphs

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Outline

Intersection graphs

- Consider a finite family $\mathcal{F}$ of non-empty sets. The intersection graph of $\mathcal{F}$ is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.
- A circular-arc graph is the intersection graph of a finite family of arcs in a circle (such a family is called a circular-arc model of the graph).

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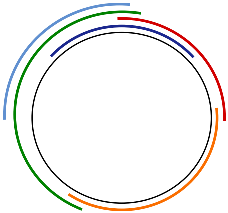

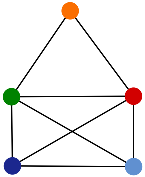

Intersection graphs

- A clique in a graph is a maximal set of pairwise adjacent vertices.
- The clique graph $K(G)$ of a graph G is the intersection graph of its cliques.

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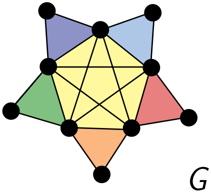

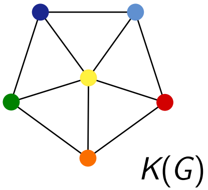

The Helly property

- A family of sets $\mathcal{F}$ is said to satisfy the Helly property if every subfamily of $\mathcal{F}$, consisting of pairwise intersecting sets, has a common element.
- A graph is clique-Helly (CH) if its cliques satisfy the Helly property, and it is hereditary clique-Helly (HCH) if all its induced subgraphs are clique-Helly.
- A graph is Helly circular-arc (HCA) if it admits a circular-arc model whose arcs satisfy the Helly property. Helly circular-arc graphs have polynomial time recognition (Gavril, 1974).

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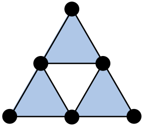

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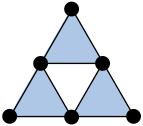
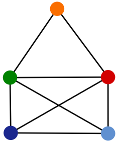
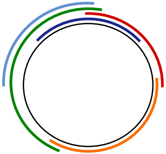
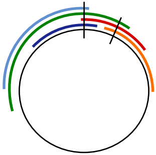
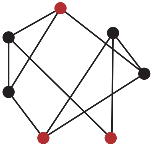





Perfect graphs

- A stable set in a graph G is a subset of pairwise non-adjacent vertices of G. The stability number $\alpha(G)$ is the cardinality of a maximum stable set of G.
- A clique cover of a graph G is a subset of cliques covering all the vertices of G. The clique-covering number $k(G)$ is the cardinality of a minimum clique cover of G.
- A graph G is perfect when $\alpha(H) = k(H)$ for every induced subgraph H of G.



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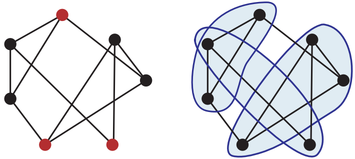

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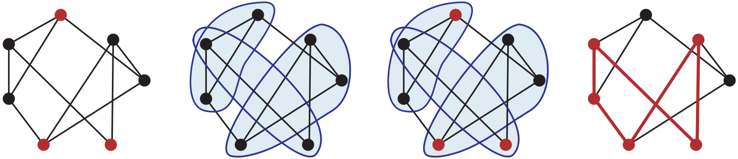
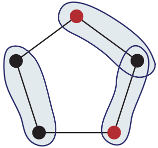
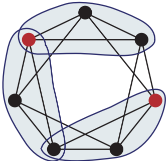


Perfect graphs

- A hole is a chordless cycle of length at least four.
- Odd holes $C_{2k+1}, k \geq 2$, are not perfect: $\alpha(C_{2k+1}) = k$ and $k(C_{2k+1}) = k + 1$.
- Complements of odd holes (odd antiholes) $\overline{C_{2k+1}}, k \geq 2$, are not perfect: $\alpha(\overline{C_{2k+1}}) = 2$ and $k(\overline{C_{2k+1}}) = 3$.



C_5



$\overline{C_7}$

Perfect graphs

Perfect graphs were introduced by Berge in 1960. He conjectured that the only minimal imperfect graphs are odd holes and their complements. Finally, this conjecture was proved by Chudnovsky, Robertson, Seymour and Thomas in 2002.

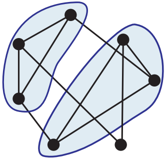
Strong Perfect Graph Theorem

A graph is perfect if and only if it neither contains an odd hole nor an odd antihole as an induced subgraph.

Perfect graphs have a polynomial time recognition algorithm (Chudnovsky, Cornujols, Liu, Seymour and Vušković, 2003).

Clique-perfect graphs

- A clique-independent set is a collection of pairwise vertex-disjoint cliques. The clique-independence number $\alpha_c(G)$ is the size of a maximum clique-independent set of G.
- A clique-transversal of a graph G is a subset of vertices that meets all the cliques of G. The clique-transversal number $\tau_c(G)$ is the size of a minimum clique-transversal of G.
- A graph G is clique-perfect when $\alpha_c(H) = \tau_c(H)$ for every induced subgraph H of G.



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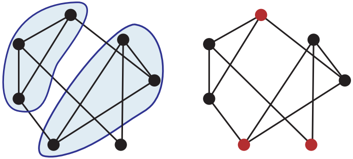

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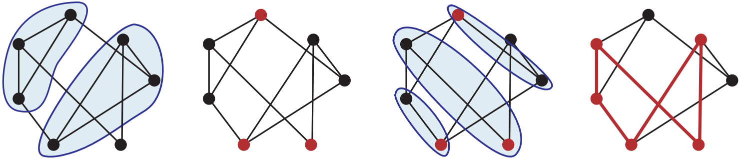

Clique-perfect graphs

- The terminology "clique-perfect" has been introduced by Guruswami and Pandu Rangan in 2000, but the equality of the parameters α_c and τ_c was previously studied by Berge in the context of balanced hypergraphs.
- The complete list of minimal clique-imperfect graphs is still not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem.
- In this work, we present a partial result in this direction, that is, we characterize clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph belongs to a certain class. Some of this characterizations lead to polynomial time recognition algorithms for clique-perfection within these classes.

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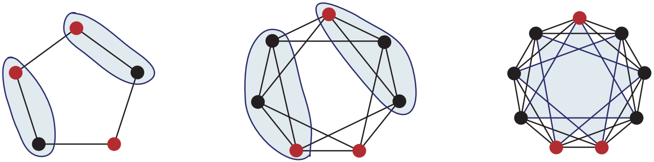
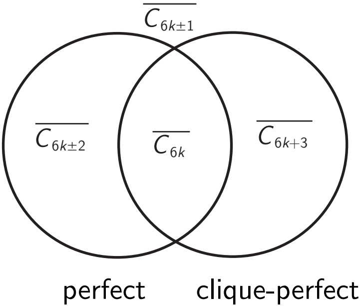
First question: is there some relation between clique-perfect graphs and perfect graphs?

- Odd holes $C_{2k+1}, k \geq 2$, are not clique-perfect: $\alpha_c(C_{2k+1}) = k$ and $\tau_c(C_{2k+1}) = k + 1$.
- Antiholes $\overline{C_n}, n \geq 5$, are clique-perfect if and only if $n \equiv 0(3)$ (Reed, 2000): $\tau_c(\overline{C_n}) = 3$ and $\alpha_c(\overline{C_n}) = 2$ or 3, being 3 only if n is divisible by three.



C_5 $\overline{C_7}$ $\overline{C_9}$

So the classes overlap and we have the following scheme of relation between perfect graphs and clique-perfect graphs:



However...

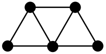
- If we look at the clique graph, it holds the following relation:
  - $\alpha_c(G) = \alpha(K(G))$.
  - $\tau_c(G) \geq k(K(G))$, and, if G is clique-Helly, $\tau_c(G) = k(K(G))$.
- A graph G is K-perfect when $K(G)$ is perfect.
- If a graph G is clique-Helly and K-perfect, then $\alpha_c(G) = \alpha(K(G)) = k(K(G)) = \tau_c(G)$.

Corollary

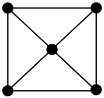
If a class of graphs is hereditary, clique-Helly and K-perfect, the class is clique-perfect.

Families of clique-perfect graphs

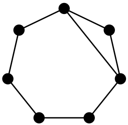
- Complements of acyclic graphs.
- Antiholes of length divisible by three.
- Comparability graphs. A graph is a comparability graph if there exists a transitive orientation of its edges.
- {3-fan,4-wheel}-free graphs such that every odd cycle has a short chord.
- Balanced graphs. A graph is balanced if its vertex-clique incidence matrix does not contain the incidence matrix of an odd cycle as a submatrix.



3-fan



4-wheel



short chord

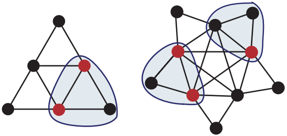
	v_1	v_2	v_3
M_1	1	1	0
M_2	0	1	1
M_3	1	0	1

Families of clique-imperfect graphs

- Odd holes.
- Antiholes of length not divisible by three.
- Odd suns.
- Odd generalized suns (they generalize odd suns and odd holes).
- Graphs S_k^1 and S_k^2, $k \geq 2$.
- Graphs Q_{6k+3}, $k \geq 0$.

Odd suns

An *r*-sun is a chordal graph with a cycle of length r and r vertices, each one of them is adjacent to the endpoints of an edge of the cycle.

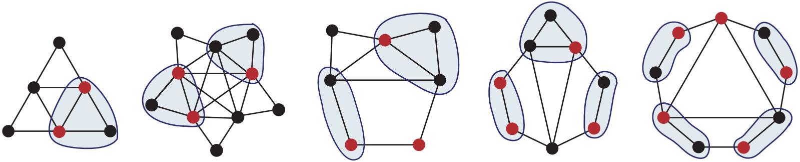


3-sun 5-sun

Odd suns are not clique-perfect: they have, as odd holes, $\alpha_c((2k+1)\text{-sun}) = k$ and $\tau_c((2k+1)\text{-sun}) = k+1$.

Odd generalized suns

A family of graphs generalizing both odd holes and odd suns are odd generalized suns (B., D., Groshaus and Szwarcfiter). An edge in a cycle is non-proper if it forms a triangle with some vertex of the cycle. An odd generalized sun is formed by an odd cycle and a vertex for each non-proper edge, adjacent only to its endpoints.



3-sun 5-sun 5-generalized sun (viking) 7-generalized sun 9-generalized sun

They have $\alpha_c((2k+1)\text{-gen. sun}) = k$ and $\tau_c((2k+1)\text{-gen. sun}) \geq k+1$.

Odd generalized suns

- With the above definition, odd generalized suns are not necessarily minimal, and it is still an open question the characterization of minimal odd generalized suns and minimally clique-imperfect odd generalized suns.
- However, in the characterization of clique-perfect HCA graphs appear only odd holes and two new kinds of minimally clique-imperfect odd generalized suns: vikings and 2-vikings.
- They can be seen in the figure, where dotted lines replace any odd induced path of length at least one.

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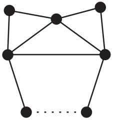
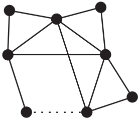


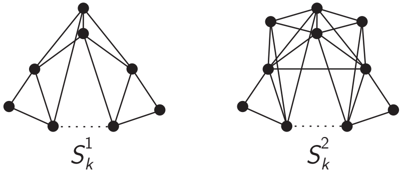
viking



2-viking

Graphs S_k^1 and S_k^2, $k \geq 2$

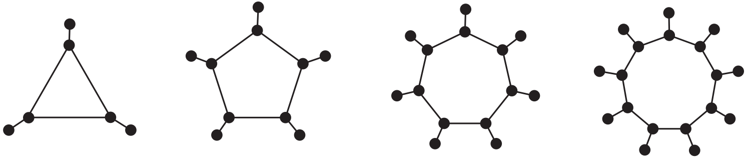
The families of graphs S_k^1 and S_k^2, $k \geq 2$, are defined based on a cycle of $2k+1$ vertices, as it can be seen in the figure, where dotted lines replace any odd induced path of length at least one.



They have $\alpha_c(S_k^i) = k$ and $\tau_c(S_k^i) = k+1$, for $i = 1, 2$.

Graphs Q_{6k+3}, $k \geq 0$

The family of graphs Q_n was defined by Szwarcfiter, Lucchesi and P. de Mello, 1998. For odd values of n, $\alpha_c(Q_n) = 1$ and $\tau_c(Q_n) = 2$ (they are exactly the graphs minimally clique-complete without a universal vertex).



$\overline{Q_3}$ $\overline{Q_5}$ $\overline{Q_7}$ $\overline{Q_9}$

But only the graphs Q_n with n odd and divisible by three are minimally clique-imperfect, the other ones contain clique-imperfect antiholes.

Chordal graphs

A graph is chordal when every cycle of length at least four has a chord. Chordal graphs have polynomial time recognition (Rose, Tarjan and Lueker, 1976).

Theorem (Lehel and Tuza, 1986)

Let G be a chordal graph. Then the following are equivalent:

1. G does not contain odd suns.
2. G is balanced.
3. G is clique-perfect.

The recognition of clique-perfect chordal graphs can be reduced to the recognition of balanced graphs, which is solvable in polynomial time (Conforti, Cornuéjols and Rao, 1999).

Line graphs

Let H be a graph. Its line graph $L(H)$ is the intersection graph of the edges of H. A graph G is a line graph if there exists a graph H such that $G = L(H)$. Line graphs have polynomial time recognition (Lehot, 1974).

Theorem

Let G be a line graph. Then the following are equivalent:

1. no induced subgraph of G is and odd hole, or a 3-sun.
2. G is clique-perfect.
3. G is perfect and it does not contain a 3-sun.

The recognition of clique-perfect line graphs can be reduced to the recognition of perfect graphs with no 3-sun, which is solvable in polynomial time.

Sketch of proof

Theorem

Let G be a line graph. Then the following are equivalent:

1. no induced subgraph of G is and odd hole, or a 3-sun.
2. G is clique-perfect.
3. G is perfect and it does not contain a 3-sun.

$1 \Leftrightarrow 3$ is a corollary of SPGT, because line graphs cannot contain antiholes $\overline{C_n}$ with $n \geq 7$ as induced subgraphs.
$2 \Rightarrow 1$ is easy.
To prove $1 \Rightarrow 2$, we prove that line graphs with neither 3-sun nor odd holes are K-perfect. Then the result is proved by induction, taking as a basic case when the graph is hereditary clique-Helly.

Claw-free hereditary clique-Helly graphs

A claw is the graph $K_{1,3}$. A graph is claw-free if it does not contain a claw as induced subgraph.

Theorem

Let G be a hereditary clique-Helly claw-free graph. Then the following are equivalent:

1. no induced subgraph of G is an odd hole, or $\overline{C_7}$.
2. G is clique-perfect.
3. G is perfect.

The recognition of clique-perfect HCH claw-free graphs can be reduced to the recognition of perfect graphs, which is solvable in polynomial time.

Sketch of proof

Theorem

Let G be a hereditary clique-Helly claw-free graph. Then the following are equivalent:

1. no induced subgraph of G is an odd hole, or $\overline{C_7}$.
2. G is clique-perfect.
3. G is perfect.

$1 \Leftrightarrow 3$ is a corollary of SPGT, because HCH graphs cannot contain antiholes $\overline{C_n}$ with $n \geq 9$ as induced subgraphs.
$2 \Rightarrow 1$ is easy.
To prove $1 \Rightarrow 2$, we prove that HCH claw-free graphs with neither $\overline{C_7}$ nor odd holes are K-perfect. The proof is by induction, based on claw-free graphs decomposition theorems (C. and Seymour).

Diamond-free graphs

A diamond is the graph $K_4 - \{e\}$, where e is any edge of K_4. A graph is diamond-free if it does not contain a diamond as induced subgraph.

Theorem

Let G be a diamond-free graph. Then the following are equivalent:

1. G contains no odd generalized sun.
2. G is clique-perfect.

Diamond-free odd generalized suns are odd generalized suns without non-proper edges. In this case, the characterization is not formulated by minimal subgraphs yet.

Sketch of proof

Theorem

Let G be a diamond-free graph. Then the following are equivalent:

1. G contains no odd generalized sun.
2. G is clique-perfect.

$2 \Rightarrow 1$ is easy.
To prove $1 \Rightarrow 2$, we prove that diamond-free graphs with no odd generalized suns are K-perfect. It remains only to observe that diamond-free graphs are hereditary clique-Helly.

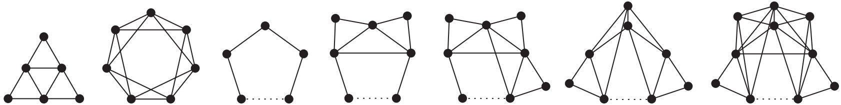
Helly circular-arc graphs

Recall that a graph G is HCA if there exists a family of arcs of a circle verifying the Helly property and such that G is the intersection graph of this family.

Theorem

Let G be a Helly circular-arc graph. Then the following are equivalent:

1. G does not contain any of the graphs in the figure, where the dotted lines replace any odd induced path of length at least one.
2. G is clique-perfect.



Sketch of proof

Theorem

Let G be a Helly circular-arc graph. Then the following are equivalent:

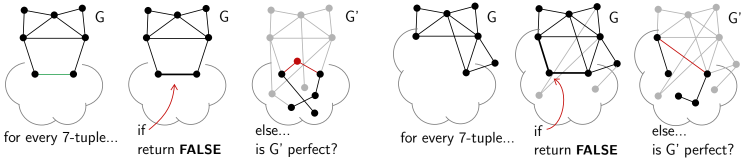
1. G does not contain any of the graphs in the figure, where the dotted lines replace any odd induced path of length at least one.
2. G is clique-perfect.

$2 \Rightarrow 1$ is easy.
To prove $1 \Rightarrow 2$, we show that Helly circular-arc graphs which do not contain the graphs of the figure as induced subgraphs are K-perfect. This is the hardest part of the proof. The remaining part is based in the fact that Helly circular-arc graphs that are not HCH have $\alpha_c = \tau_c$ or they are clique-complete without a universal vertex, and then we use a characterization of clique-complete graphs by Szwarcfiter, Lucchesi and P. de Mello, 1998.

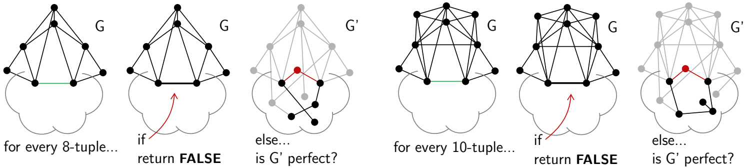
Recognition algorithm

Input: A *HCA* graph G; **Output: TRUE** if G is clique-perfect and **FALSE** if G is not.

1. Check if G contains a 3-sun. Case yes, return **FALSE**.
2. Check for odd holes and $\overline{C_7}$: check if G is perfect. Case not, return **FALSE**.
3. Check for vikings and 2-vikings:



4. Check for S_k^1 and S_k^2:



5. If no forbidden subgraph is found, return **TRUE**.

Summary

Class	Forbidden induced subgraphs	Recognition
Chordal	odd suns	**P**
Line graphs	odd holes, 3-sun	**P**
HCH claw-free	odd holes, $\overline{C_7}$	**P**
Diamond-free	odd generalized suns	**?**
HCA	3-sun, odd holes, $\overline{C_7}$, vikings, 2-vikings, S_k^1, S_k^2	**P**