On coloring problems with local constraints

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Abstract

We show complexity results for some generalizations of the graph coloring problem on two classes of perfect graphs, namely clique trees and unit interval graphs. We deal with the μ-coloring problem (upper bounds for the color on each vertex), the precoloring extension problem (a subset of vertices colored beforehand), and a problem generalizing both of them, the (γ, μ)-coloring problem (lower and upper bounds for the color on each vertex). We characterize the complexity of all those problems on clique trees of different heights, providing polytime algorithms for the cases that are easy. These results have two interesting corollaries: first, one can observe on clique trees of different heights the increasing complexity of the chain k-coloring, μ-coloring, (γ, μ)-coloring, list-coloring. Second, clique trees of height 2 are the first known example of a class of graphs where μ-coloring is polynomial time solvable and precoloring extension is NP-complete, thus being at the same time the first example where μ-coloring is polynomially solvable and (γ, μ)-coloring is NP-complete. Last, we show that the μ-coloring problem on unit interval graphs is NP-complete. These results answer three questions from [3].

Keywords: clique trees, computational complexity, graph coloring problems, unit interval graphs

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1 Introduction

A *coloring* of a graph $G = (V, E)$ is a function $f : V \to \mathbb{N}$ such that $f(v) \neq f(w)$ whenever $vw \in E$. A *k-coloring* is a coloring f such that $f(v) \leq k$ for every $v \in V$. The *vertex coloring problem* takes as input a graph G and a natural number k, and consists in deciding whether G is k-colorable or not. This well-known problem is a basic model for scheduling, frequency assignment and resource allocation problems.

In order to take into account particular constraints arising in practical settings, more elaborate models of vertex coloring have been defined in the literature. A hierarchy of such models was studied in [3]. Two generalizations of the k-coloring problem are precoloring extension [1] and μ-coloring [2].

The *precoloring extension* (PrExt) problem takes as input a graph $G = (V, E)$, a subset $W \subseteq V$, a coloring f' of W, and a natural number k, and consists in deciding whether G admits a k-coloring f such that $f(v) = f'(v)$ for every $v \in W$ or not. In other words, a prespecified vertex subset is colored beforehand, and the goal is to extend this partial coloring to a valid k-coloring of the whole graph.

Given a graph $G(V, E)$ and a function $\mu : V \to \mathbb{N}$, the *μ-coloring* problem consists in deciding whether G is *μ-colorable*, i.e. whether there exists a function $f : V \to \mathbb{N}$ such that $f(v) \leq \mu(v)$ for every $v \in V$ and f is a k-coloring of G for some natural k. This model arises in the context of classroom allocation to courses, where each course must be assigned to a classroom which is large enough to accommodate the students taking the course.

A problem generalizing the latter two problems is the (γ, μ)-coloring problem [3], where also lower bounds for the color of each vertex are determined: given a graph $G(V, E)$ and functions $\gamma, \mu : V \to \mathbb{N}$ such that $\gamma(v) \leq \mu(v)$ for every $v \in V$, the *(γ, μ)-coloring* problem consists in deciding whether there exists a μ-coloring f where additionally $\gamma(v) \leq f(v)$ for every $v \in V$.

Finally, a model generalizing all of the previous problems is the *list-coloring problem*, which considers a prespecified set of available colors for each vertex. Given a graph G and a finite list $L(v) \subseteq \mathbb{N}$ for each vertex $v \in V$, the list-coloring problem asks for a *list-coloring* of G, i.e., a coloring f such that $f(v) \in L(v)$ for every $v \in V$. This problem is NP-complete even in very restricted classes of graphs, like complete bipartite or complete split graphs [12], but it is polynomial time solvable on block graphs (i.e. graphs whose 2-connected

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components are cliques), by combining dynamic programming techniques and a maximum matching algorithm [11].

The scheme of generalizations imply that all the problems in this hierarchy are polynomially solvable in those graph classes where list-coloring is polynomial and, on the other hand, all the problems are NP-complete in those graph classes where vertex coloring is NP-complete.

The complexity of this family of problems over different classes of graphs was studied, and there are several examples of classes where k-coloring is polynomial-time solvable but precoloring extension and μ-coloring are NP-complete, like bipartite graphs [2,8], interval graphs [1,3,5] and distance hereditary graphs [3,7], and where (γ, μ)-coloring is polynomial time solvable but list-coloring is NP-complete, like complete bipartite graphs and complete split graphs [3,12]. The problems of precoloring extension and μ-coloring are not directly related, i.e., no one is a generalization of the other one, and the reductions among them involve changes in the input graph. Nevertheless, for almost all the graph classes where their complexity is known, they are in the same side of the dichotomy "polynomial time solvable vs. NP-complete". The class of split graphs is the only known exception where precoloring extension is polynomial time solvable [9] while μ-coloring is NP-complete [3], while, to the best of our knowledge, so far no class of graphs where μ-coloring is polynomially solvable and precoloring extension is NP-complete was known. In fact, not even a class where (γ, μ)-coloring is NP-complete while μ-coloring is polynomially solvable was known so far, and these are mentioned as open problems in [3].

In this work we study the complexity of these coloring problems on two classes of perfect graphs, namely clique trees and unit interval graphs. We characterize the complexity of all those problems on clique trees of different heights, providing polytime algorithms for the cases that are easy. These results have two interesting corollaries: first, one can observe on clique trees of different heights the increasing complexity of the chain k-coloring, μ-coloring, (γ, μ)-coloring, list-coloring. Second, clique trees of height 2 are the first known example of a class of graphs where μ-coloring is polynomial time solvable and precoloring extension is NP-complete, thus being at the same time the first example where μ-coloring is polynomially solvable and (γ, μ)-coloring is NP-complete; this solves two questions from [3]. Last, we show that the μ-coloring problem on unit interval graphs is NP-complete, thus settling another question from [3].

2 Notation and main results

Two vertices of a graph are *true twins* if they are adjacent and they have the same neighbors. The *multiplicity* of a vertex is the number of its true twins, including the vertex itself. A *clique tree* is a graph G such that the graph obtained by identifying true twins is a tree, which is called *basic tree*. The height of a rooted tree is the maximum distance from the root to any other vertex of the tree. The height of a tree is the minimum height over all the choices of a root. The *height* of a clique tree G is the height of its basic tree.

A graph G is an *interval graph* (resp. *circular-arc graph*) if it is the intersection graph of a set of intervals over the real line (resp. the unit circle). A *unit interval graph* is the intersection graph of a set of intervals of length one, while a *proper interval graph* is the intersection graph of a set of intervals where no interval is properly contained in another. It is known that the classes of unit interval graphs and proper interval graphs coincide [13].

2.1 Clique trees

The k-coloring problem can be solved in polynomial time for clique trees, since they are perfect graphs. Moreover, the size of the maximum clique is the maximum sum of the multiplicities of two adjacent vertices that are not twins. The list-coloring problem can be solved in polynomial time on clique trees of height zero (i.e for complete graphs), since it can be modelled as a maximum matching problem on a bipartite graph. Meanwhile, it is known that this problem is NP-complete for clique trees of height 1, even if its basic tree is formed by a root and two children [10], or if the multiplicity of all the vertices but the root is 1 [12]. These results are tight, since every basic tree has either one or at least three vertices, and if the multiplicity of all vertices is one, the clique tree is indeed a tree and thus a block graph. Next we state the results obtained for precoloring extension, μ-coloring and (γ, μ)-coloring.

Theorem 2.1 *The (γ, μ)-coloring problem can be solved in polynomial time for clique trees of height at most* 1*.*

The algorithm is based on an integer programming formulation with a totally unimodular constraint matrix, and uses a lemma which gives a condition similar to Hall's matching condition.

Theorem 2.2 *The μ-coloring problem can be solved in polynomial time for clique trees of height at most* 2*.*

The algorithm consists on a polynomial time reduction from the μ-coloring problem on clique trees of height 2 to the (γ, μ)-coloring problem on clique trees of height 1.

Theorem 2.3 *For each fixed $p \geq 2$, the precoloring extension problem is NP-complete on clique trees of height p.*

Corollary 2.4 *For each $p \geq 2$ the (γ, μ)-coloring problem is NP-complete on clique trees of height p.*

Theorem 2.5 *For each $p \geq 3$, the μ-coloring problem is NP-complete on clique trees of height p.*

The two NP-completeness proofs are based on reductions from 3-SAT.

Let us remark that clique trees of different heights constitutes a class of graphs where we can observe the different complexity of the hierarchy of problems k-coloring, μ-coloring, (γ, μ)-coloring, list-coloring (cfr. Table 2.1); furthermore, the class of clique trees of height 2 is, to the best of our knowledge, the first known example where μ-coloring is solvable in polynomial time, while precoloring extension (and thus (γ, μ)-coloring) is NP-complete.

Problem	Height			
	0	1	2	fixed $p \geq 3$
k-coloring	P	P	P	P
μ-coloring	P	P	P	NP-c
precoloring	P	P	NP-c	NP-c
(γ, μ)-coloring	P	P	NP-c	NP-c
list-coloring	P	NP-c	NP-c	NP-c

Table 1
A summary of the complexity results for clique trees.

2.2 Unit interval graphs

The μ-coloring problem was motivated by a scheduling problem [2], so one of the classes of interest for this problem is the class of interval graphs and, in particular, the class of unit interval graphs. In [3], it is proved that the μ-coloring problem is NP-complete on interval graphs. The proof is based on a reduction from k-coloring on circular-arc graphs, which is NP-complete [4]. But this proof cannot be reproduced for unit interval graphs, since the k-coloring problem on proper circular-arc graphs is polynomial-time solvable [14]. In fact, in [3] the complexity of μ-coloring problem on unit interval graphs is posted as an open problem. In this work we settle this question.

Theorem 2.6 *The μ-coloring problem is NP-complete on unit interval graphs.*

The reduction is quite involved and uses Restricted 3-SAT, whose input is a Boolean formula in conjunctive normal form, where each variable appears at most three times, each literal appears at most twice, each clause contains at most three literals, and the goal is to determine if there exists a satisfying truth assignment for it. This problem is known to be NP-complete, see e.g. [6].

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Note: A preliminary version of the full article is available on http://mate.dm.uba.ar/~fbonomo/files/BFO-completo09_v5.pdf, just for refereeing purposes.