Bounded coloring of co-comparability graphs and the pickup and delivery tour combination problem

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Abstract

The Double Traveling Salesman Problem with Multiple Stacks is a vehicle routing problem in which pickups and deliveries must be performed in two independent networks. The items are stored in stacks and repacking is not allowed. Given a pickup and a delivery tour, the problem of checking if there exists a valid distribution of items into s stacks of size h that is consistent with the given tours, is known as Pickup and Delivery Tour Combination (PDTC) problem.

In the paper, we show that the PDTC problem can be solved in polynomial time when the number s of stacks is fixed but the size of each stack is not. We build upon the equivalence between the PDTC problem and the bounded coloring (BC) problem on permutation graphs: for the latter problem, s is the number of colors and h is the number of vertices that can get a same color. We show that the BC problem can be solved in polynomial time when s is a fixed constant on co-comparability graphs, a superclass of permutation graphs. To the contrary, the BC problem is known to be hard on permutation graphs when $h \geq 6$ is a fixed constant, but s is unbounded.

Key words: bounded coloring, capacitated coloring, equitable coloring, permutation graphs, scheduling problems, thinness

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1. Introduction

The growing complexity of planning strategies in transportation and logistic companies has recently induced the appearance of new vehicle routing problems. One of these problems is the Double Traveling Salesman Problem with Multiple Stacks (DTSPMS), introduced by Petersen and Madsen [31]. It is a routing problem in which some pickups and deliveries must be performed in two independent networks, verifying some precedence and loading constraints imposed on the vehicle. Two independent regions that are supposed to be very far apart are considered, and n items must be picked up in different locations of the first region and delivered to the corresponding locations of the second region. A single vehicle is used and repacking is not allowed. The load can be packed in several compartments that must obey the Last-In-First-Out (LIFO) principle, while there are no mutual constraints between two different compartments. From now on the compartments of the container will be referred to as *stacks*. We will denote by s the number of stacks and by h the maximum number of items that can be placed into each stack. In the real-life application that originally motivated the problem, the items to be delivered were standardized Euro Pallets, that could be used to load different kinds of goods. The vehicles used to carry the goods had a 40-foot pallet container in which standardized Euro Pallets fit 3 by 11, suggesting the use of 3 available stacks with a maximum capacity of 11 items. The driver was not allowed to touch the goods due to security and insurance reasons and, as a consequence, repacking was not permitted during the whole process. Hence, a feasible solution for an instance of the DTSPMS consists of a *pickup tour*, a *delivery tour* and a *stack assignment*, that is, which stack is assigned to each item. The goal is to minimize the sum of the lengths of the pickup and delivery tours. Some heuristics and exact methods have been presented to solve the DTSPMS [30, 12, 13, 25].

One of the strategies that has been proposed [25] consists in considering the k best solutions to the TSP problem on each network (for some value of k), thus generating k possible pickup (resp. delivery) tours, and then checking for which pairs of tours there exists a feasible stack assignment. The subproblem of checking whether, for a given pickup tour and a given delivery tour, there exists a feasible stack assignment is known as the *pickup and delivery tour combination* (PDTC) problem. Lusby et al. [25] solve the PDTC problem by an integer programming model. Casazza et al. [8] observe that the problem can be solved in polynomial time if the capacity constraints on each stack are relaxed, since, in this case, the problem reduces to the graph coloring problem on permutation graphs, that is known to be poly-time solvable [32].

In this paper we concentrate on the PDTC problem, and show that it can be solved in polynomial time, when the number of stacks is a fixed constant. As we mentioned before, in the real case that originally motivated the problem, $s = 3$ and $h = 11$. So, from the practical point of view, it is reasonable to treat s as a constant and h as a parameter.

In order to achieve this result, we follow the graph coloring approach in [8], but take into account the capacity constraints. We therefore show that the PDTC problem is equivalent to a capacitated coloring problem, the *bounded coloring* (BC) problem, on permutation graphs. We then show that the *capacitated coloring* (CC) problem, a problem that generalizes the BC problem – as well as another coloring problem, the *equitable coloring* problem – can be solved in polynomial time on *co-comparability graphs*, a superclass of permutation graphs, when the maximum number of colors s is fixed.

Our proof builds upon some tools developed in [26]. In that paper, a graph invariant, the *thinness* of a graph, is introduced. Graphs with bounded thinness are a generalization of *interval graphs*, that are exactly the graphs with thinness 1. In [26] it is shown that the maximum weighted stable set problem can be solved in polynomial time on graphs with bounded thinness, when a suitable representation $\mathcal{R}$ of the graph is given. We show in this paper that, in the same hypotheses, also the CC problem can be solved in polynomial time, if the number of colors s is fixed. We then make use of the following crucial fact: given an instance of the CC problem on a co-comparability graph G, if the number of colors s fixed, then, in polynomial time, we may either show that the instance is infeasible, or show that G has thinness at most s, and in this case provide the representation $\mathcal{R}$ for G.

We point out the following interesting fact. The bounded coloring problem is known to be hard on permutation graphs [24] and interval graphs [6], if the (maximum) number h of vertices that can get the same color is a fixed constant, but the (maximum) number s of colors is unbounded. Our results show that, to the contrary, the problem is poly-time solvable, on the same classes of graphs, if s is a fixed constant, while h is unbounded. Given the equivalence between the BC problem on permutation graphs and the PDTC problem, the same fact holds for the latter problem.

In Section 2 we formally define the pickup and delivery tour combination problem, as well as the coloring problems we are interested in. In Section 3 we recall the definition of co-comparability graphs and graphs with bounded thinness, and show that the thinness of a co-comparability graph is bounded by its chromatic number. In Section 4 we deal with the capacitated coloring problem on graphs with bounded thinness.

Unless otherwise specified, in the paper we deal with simple and undirected graphs. Let $G(V, E)$ be such a graph. The complement of G is denoted as $\overline{G}$, while $G[S]$ denotes the subgraph induced by a set $S \subseteq V$. We denote by $N(v)$ the neighborhood of a vertex $v \in V$, i.e., the set of vertices that are adjacent to v.

2. The Pickup and Delivery Tour Combination Problem

Suppose that we are given a pickup and a delivery tour for some instance of the DTSPMS problem, with s stacks of size (height) h. In particular, let $1, \ldots, n$ be the list of items in the reverse order in which they should be picked up, i.e., the item 1 is the last item picked up while the item n is the first one. Also let the permutation π of $\{1, \ldots, n\}$ give the order in which items should be delivered, namely, if $\pi(i) < \pi(j)$, then item j has to be delivered after item i. Thus, two items i, j, with $1 \leq i < j \leq n$, can be placed in the same stack if and only if $\pi(i) < \pi(j)$.

Pickup and Delivery Tour Combination (PDTC) problem is then the following problem: **Given** the delivery permutation π, the maximum number of items per stack h, the number of stacks s. **Find** an assignment $\varphi : \{1, \ldots, n\} \mapsto \{1, \ldots, s\}$ of items to stacks such that:

(coloring constraint) for each pair of items $\{i, j\}$ assigned to the same stack, i.e., $\varphi(i) = \varphi(j)$, if $i < j$, then $\pi(i) < \pi(j)$;

(capacity constraint) no more than h items are assigned to each stack.

In the following, we specify an instance of the PDTC problem by a triple (π, h, s), where π is a permutation of $\{1, \ldots, n\}$ and h, s are positive integers.

Recall that we may associate to each permutation of $\{1, \ldots, n\}$ a graph $G(\pi)$ with vertices $\{v_1, \ldots, v_n\}$ and such that v_i and v_j, with $1 \leq i < j \leq n$, are adjacent if and only if $\pi(i) > \pi(j)$. Such a graph is called a *permutation graph*, and more in general we have the following definition.

Definition 1. *A graph $G(V, E)$ is a* permutation graph *if there is an order $1, \ldots, n$ of V and a permutation π such that, for $1 \leq i < j \leq n$, $ij \in E$ if and only if $\pi(i) > \pi(j)$.*

Permutation graphs were introduced in [14, 32], and different techniques for solving algorithmic problems on a permutation graph are given in [7].

Going back to the PDTC problem, in [8] Casazza et al. show that, if for some instance (π, h, s) of the problem we relax the capacity constraints, then the problem reduces to finding an s-coloring on $G(\pi)$.

Definition 2. *An* s-coloring *of a graph* $G(V, E)$ *is a mapping* $\varphi : V \to \{1, \ldots, s\}$ *such that if* $uv \in E$ *then* $\varphi(u) \neq \varphi(v)$*.*

In fact, an s-coloring φ of $G(\pi)$ is also an assignment of items to stacks that satisfies the coloring constraint, and, vice versa, an assignment φ of item to stacks, that satisfies the coloring constraint, is also an s-coloring of $G(\pi)$. Trivially, if we do *not* relax the capacity constraint, we may then associate to each instance of the PDTC problem an instance of the:

Bounded Coloring (BC) problem. Given a graph $G(V, E)$ and integers $s, h \geq 1$. **Find** a *capacitated s-coloring*, i.e., an s-coloring φ of G such that the number of vertices taking color i is bounded by h, i.e., $|\varphi^{-1}(i)| \leq h$, for each $i = 1..s$.

An instance of the bounded coloring problem is described by a triple (G, h, s). Then we may solve an instance (π, h, s) of the PDTC problem, by finding a capacitated s-coloring on the instance $(G(\pi), h, s)$ of the BC problem. However, one should observe that also one may solve an instance $(G(\pi), h, s)$ of the BC problem (on a permutation graph) by solving the instance (π, h, s) of the PDTC problem. Therefore, the PDTC problem is *equivalent* to the BC problem on permutation graphs.

We recall that while the coloring problem, i.e., the problem of finding an s-coloring of a given graph with minimum s, can be solved in $O(n \log n)$ on permutation graphs with n vertices [32], Jansen showed that the bounded coloring problem is NP-hard on permutation graphs for any fixed $h \geq 6$ [19]. We may therefore state the following corollary of Jansen's result:

Corollary 3. *The PDTC problem is NP-hard, even if the size* h *of each stack is a fixed constant (greater than 5).*

In the following we focus on what happens to the BC problem on permutation graphs (and therefore to the PDTC problem) when we instead fix the number s of colors, while leaving unbounded h.

We indeed deal with a slight *generalization* of the BC problem on a *superclass* of permutation graphs. First, we consider the following:

Capacitated Coloring (CC) problem. **Given** a graph $G(V, E)$, an integer $s \geq 1$ and capacities $(\alpha_1^*, \ldots, \alpha_s^*) \in \mathbb{Z}_+^s$. **Find** a *capacitated s-coloring*, i.e., an s-coloring φ of G such that the number of vertices taking color i is bounded by α_i^*, i.e., $|\varphi^{-1}(i)| \leq \alpha_i^*$, for each $i = 1..s$.

An instance of the *capacitated coloring problem* is described by a triple (G, α^*, s). Note that the CC problem generalizes the BC problem, arising when $\alpha_1^* = \ldots = \alpha_s^* = h$.

Then, we deal with a superclass of permutation graphs, that of comparability graphs:

Definition 4. *A graph $G(V, E)$ is a* comparability graph *if there exists an ordering $v_1, \ldots, v_n$ of V such that, for each triple (p, q, r) with $p < q < r$, if $v_p v_q$ and $v_q v_r$ are edges of G, then so is $v_p v_r$. Such an ordering is a* comparability ordering.

Comparability graphs can be recognized in linear time [27]. Moreover, the recognition algorithm outputs a comparability ordering for the graph. A *co-comparability graph* is the complement of a comparability graph. Permutation graphs are exactly the graphs which are comparability and co-comparability graphs [11]. In fact, it is easy to see that the ordering $1, \ldots, n$ in the Definition 1 is a comparability ordering for both G and its complement.

It follows from above that the PDTC problem is a specialization of the CC problem on co-comparability graphs. On this class of graphs, while the coloring problem can be solved in $O(n^3)$ [15] (where n is the number of vertices), the BC problem is NP-hard, for any fixed $h \geq 3$ [24]. Our main result is Theorem 12, showing that, vice versa, the problem can be solved in polynomial time if h is unbounded but s is fixed.

2.1. Related literature on the bounded coloring problem

The BC problem has a number of applications. It is also known in the literature as *mutual exclusion scheduling* (MES) [2], which is the problem of scheduling unit-time tasks non-preemptively on h processors subject to constraints, represented by a graph G. The scheduling must be such that tasks corresponding to adjacent vertices must run in disjoint time intervals. A schedule of length s corresponds to a bounded coloring of G. This problem arises in load balancing for the parallel solution of partial differential equations by domain decomposition [2, 33]. Problems of this form have been studied in [3, 23]. Other applications are in scheduling of communication systems [18] and in constructing school timetables [22]. Bodlaender and Jansen [5] studied the decision problem of a complementary scheduling problem. Their problem is similar to MES, but if two tasks are adjacent in G then they cannot be executed on the same processor. Therefore, in MES an independent set is processed in a time unit, whereas in compatibility scheduling an independent set is executed on one processor.

The BC problem can be solved in linear time on trees [2, 20] and in polynomial time on split graphs, complements of interval graphs [6, 9, 24], complements of bipartite graphs [6], and graphs with bounded treewidth [4]. Moreover, a linear time algorithm was proposed in [21] when restricted to graphs with bounded treewidth and fixed h. For fixed h or s the problem is polynomial time solvable on cographs [6, 24], for fixed h on bipartite graphs [6, 17] and line graphs [1], and for fixed s on interval graphs [6]. For $h = 2$ the problem is equivalent to the maximum matching problem on the complement graph and, therefore, is polynomial. The problem remains NP-complete on cographs, bipartite and interval graphs [6], on co-comparability graphs and fixed $h \geq 3$ [24], on complements of line graphs and fixed $h \geq 3$ [10]. For fixed $s \geq 3$ the problem is NP-complete on bipartite graphs [6].

2.2. The equitable coloring problem

The CC problem also generalizes another known coloring problem:

Equitable Coloring problem. [28, 16] Given a graph $G(V, E)$ and an integer $s \geq 1$. **Find** an *equitable s-coloring*, i.e., an s-coloring φ of G such that, for each $i, j \in \{1, \ldots, s\}$, $||\varphi^{-1}(i)| - |\varphi^{-1}(j)|| \leq 1$.

Let $r = |V| \mod s$. By setting $\alpha_1^*, \ldots, \alpha_r^* = \lceil \frac{|V|}{s} \rceil$ and $\alpha_{r+1}^*, \ldots, \alpha_s^* = \lfloor \frac{|V|}{s} \rfloor$, equitable coloring can be reduced to capacitated coloring.

Almost all complexity results for the BC problem on different graph classes mentioned in Section 2.1 can also be obtained for the equitable coloring problem by making use of the following observations: a graph G on n vertices is h-bounded s-colorable if and only if the graph G' obtained from G by adding an independent set of size $hs - n$ is equitably s-colorable; conversely, a graph G with n vertices is equitably s-colorable if and only if the graph G'' obtained from G by adding a clique of size $\lceil \frac{n}{s} \rceil s - n$ is $\lceil \frac{n}{s} \rceil$-bounded s-colorable.

3. *K*-thin graphs

The proof of our results builds upon some tools developed in [26]. We start this section with recalling some definitions and facts from that paper.

Definition 5. *A graph $G(V,E)$ is k-*thin *if there exist an ordering $v_1,\dots,v_n$ of V and a partition of V into k classes $(V^1,\dots,V^k)$ such that, for each triple (p,q,r) with $p<q<r$, if v_p, v_q belong to the same class and $v_rv_p \in E$, then $v_rv_q \in E$.*

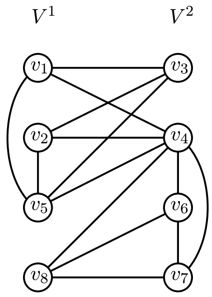

Figure 1: *A graph along with a consistent ordering and 2-partition of the vertices. The non-empty sets $N(v_r,j)_<$ are: $N(v_3,1)_< = \{v_1,v_2\}; N(v_4,1)_< = \{v_1,v_2\}; N(v_5,1)_< = \{v_1,v_2\}; N(v_5,2)_< = \{v_3,v_4\}$; $N(v_6,2)_< = \{v_4\}; N(v_7,2)_< = \{v_4,v_6\}; N(v_8,2)_< = \{v_4,v_6,v_7\}$. Then, $\Delta(1)_< = 2$, $\Delta(2)_< = 3$.*

If so, the partition and the ordering are said to be *consistent*. The minimum k such that G is k-thin is called the *thinness* of G and denoted by $\mathrm{thin}(G)$. Graphs with bounded thinness were introduced in [26] as a generalization of *interval graphs*, that are exactly the 1-thin graphs.

The thinness of a graph can be unbounded. For instance, this is the case with $\overline{tK_2}$, i.e., the complement of a matching of size t, whose thinness is t. This is shown in the following lemma, that is useful to acquaint with Definition 5.

Lemma 6. *The thinness of $\overline{tK_2}$ is t.*

Proof. Let $V(\overline{tK_2})$ be equal to $\{x_1,y_1,\dots,x_t,y_t\}$ and suppose that $\{x_i,y_i\}$, for $1 \le i \le t$, are the only pairs of non-adjacent vertices. If we define, for $1 \le i \le t$, $V^i = \{x_i,y_i\}$, then every ordering on the vertices of $V(\overline{tK_2})$ is consistent with this partition. We now show that $\overline{tK_2}$ is not $(t-1)$-thin.

Suppose the contrary, that is there exists an ordering $>$ and a partition of $V(\overline{tK_2})$ in $(t-1)$ classes $(V^1,\dots,V^{t-1})$ which are consistent. For every class V^j, denote by $f(V^j)$ the smallest element in the class with respect to the ordering $>$. By the pigeon-hole principle, there exists at least one pair $\{x_i,y_i\}$, $1 \le i \le t$, such that $\bigcup_j f(V^j) \cap \{x_i,y_i\} = \emptyset$. Without loss of generality, assume that such pair is $\{x_1,y_1\}$ and that $y_1 > x_1$. Let V^q be the class which x_1 belongs to. It follows that y_1 is adjacent to $f(V^q)$ and non-adjacent to x_1; moreover $y_1 > x_1 > f(V^q)$, a contradiction. □

Let $G(V,E)$ be a k-thin graph, and let $v_1,\dots,v_n$ and $(V^1,\dots,V^k)$ be an ordering and a partition of V which are consistent. Note that the ordering induces an order on each class V^j. For each vertex v_r and class V^j, let:

$N(v_r,j)_<$ the set of neighbors of v_r in V^j that are smaller than v_r, i.e., $N(v_r,j)_< = V^j \cap \{v_1,\dots,v_{r-1}\} \cap N(v_r)$,

and for each class V^j, let:

$\Delta(j)_<$ the maximum size of $N(v_r,j)_<$ over all vertices v_r, i.e. $\Delta(j)_< = \max_{r=1..n} |N(v_r,j)_<|$.

We want to emphasize the following fact, that gives an alternative definition for k-thin graphs and will be fundamental for the following:

Fact 7. *For each vertex $v_r \in \{v_1,\dots,v_n\}$ and each $j \in \{1,\dots,k\}$, the set $N(v_r,j)_<$ is such that:*

- *the vertices in $N(v_r,j)_<$ are consecutive, with respect to the order induced on V^j.*
- *if $N(v_r,j)_< \neq \emptyset$, then it includes the largest vertex in $V^j \cap \{v_1,\dots,v_{r-1}\}$.*

Our interest in k-thin graphs is motivated by the following facts.

Theorem 8. *Let $G(V,E)$ be a co-comparability graph. Then, thin$(G) \leq \chi(G)$. Moreover, any vertex partition given by a coloring of G and any comparability ordering for its complement are consistent.*

Proof. Let $v_1,\dots,v_n$ be a comparability order of V for $\overline{G}$. Let φ be a k-coloring of G and let $V^i = \varphi^{-1}(i)$, for $i = 1,\dots,k$. We will show that the ordering $v_1,\dots,v_n$ and the partition $V^1,\dots,V^k$ are consistent. Let (p,q,r) with $1 \leq p < q < r \leq n$, such that $v_r v_p \in E$ and v_p, v_q belong to the same (color) class, so $v_p v_q \notin E$. If $v_q v_r \notin E$, since $v_1,\dots,v_n$ is a comparability order of V for $\overline{G}$, then $v_r v_p \notin E$, a contradiction. This proves that any vertex partition given by a coloring of G and any comparability ordering for $\overline{G}$ are consistent. In particular, any partition given by an optimum coloring is consistent with $v_1,\dots,v_n$, thus thin$(G) \leq \chi(G)$. □

The bound in Theorem 8 can be arbitrarily bad: for example, if G is a clique of size n, then thin$(G) = 1$ and $\chi(G) = n$. However, it holds with equality for the graph $\overline{tK_2}$, defined in Section 5, that has thinness and chromatic number t.

Theorem 9. *Given an instance (G, α^*, s) of the capacitated coloring problem, with G a co-comparability graph with n vertices, in time $O(n^3)$ we may either decide that the instance is infeasible, or provide an ordering and an s-partition of V which are consistent.*

Proof. We evaluate $\chi(G)$ in $O(n^3)$-time [15]: if it is greater than s, the instance is infeasible. Else, by Theorem 8, G is a $\chi(G)$-thin graph, and the partition of $V(G)$ given by an optimum coloring of G is consistent with respect to a comparability ordering of its complement. □

It follows from above that our quest for a polynomial time algorithm for the CC problem with a fixed number of colors on co-comparability graphs reduces to the quest for a polynomial time algorithm for the same problem on a graph with bounded thinness and for which an ordering and a consistent partition are given.

We point out that, while the complexity of the coloring problem on graphs with bounded thinness (to the best of our knowledge) is not known, the BC problem (and, therefore, the CC problem) in graphs with bounded thinness is NP-complete, since it is NP-complete for fixed $h \geq 4$ on interval graphs [6].

4. Capacitated coloring on k-thin graphs

Consider an instance $(G(V, E), \alpha^*, s)$ of the CC problem on a graph G with n vertices that is k-thin. W.l.o.g. we assume that $\sum_{i=1..s} \alpha_i^* \geq n$. We also let $v_1, \ldots, v_n$ and $(V^1, \ldots, V^k)$ be an ordering and a partition of V which are consistent.

We reduce the CC problem on $G(V, E)$ to a reachability problem on an auxiliary acyclic digraph $D(N, A)$. Note that, while the graph G is undirected, the digraph D is directed; therefore, in order to avoid confusion we will refer to the elements of V and E as *vertices* and *edges* (as we did so far), and to the elements of N and A as *nodes* and *arcs*.

The digraph D will be *layered*, i.e., the set N is the disjoint union of subsets (layers) $N_0, N_1, \ldots, N_n$ and all edges of A have the form (u, w) with $u \in N_r$ and $w \in N_{r+1}$, for some $0 \leq r \leq n-1$. Note that there is a layer N_r, $r \neq 0$, for each vertex $v_r \in V$: we denote by $j(r)$ the class index q such that $v_r \in V^q$. Let $S = \{1, \ldots, s\}$.

We first describe the set of nodes in each layer. The first layer has only s nodes i.e., $N_0 = \{z_1, \ldots, z_s\}$. As for layers $N_1, \ldots, N_{n-1}$, there is a one-to-one correspondence between nodes at layer N_r and $(s(k+1)+1)$-tuples

$(r, \{\alpha_i\}_{i=1..s}, \{\beta_i^j\}_{i=1..s,j=1..k})$, with $0 \leq \alpha_i \leq \alpha_i^*$, for each i, and $0 \leq \beta_i^j \leq \Delta(j)_<$, for each i, j. As for the last layer, it has only one node t corresponding to the tuple $(n, \{\alpha_i^*\}_{i=1..s}, 0, \ldots, 0)$.

We associate with each node $u \notin N_0$ a suitable capacitated s-coloring problem *with additional constraints*, that we call the *constrained sub-problem* associated with u. As we show in the following, u is reachable from a node $z \in N_0$ if and only if this constrained sub-problem has a solution. Namely, we will show that the following property holds:

($*$) a node $(r, \{\alpha_i\}_{i=1..s}, \{\beta_i^j\}_{i=1..s,j=1..k})$ is reachable from a node $z \in N_0$ if and only if $G[\{v_1, \ldots, v_r\}]$ admits a capacitated s-coloring, with capacities $\alpha_1, \ldots, \alpha_s$, and with the additional constraint that, for each $i = 1..s$ and $j = 1..k$, color i is forbidden for the last β_i^j vertices in $V^j \cap \{v_1, \ldots, v_r\}$.

In this case, G admits a capacitated s-coloring with capacities $\alpha_1^*, \ldots, \alpha_s^*$ if and only if the node t is reachable from a node $z \in N_0$.

Property ($*$) will follow from a careful definition of the set of arcs A, that is given in the following. Let $u = (r, \{\alpha_i\}_{i=1..s}, \{\beta_i^j\}_{i=1..s,j=1..k})$. Note that the problem associated with u has a solution where the vertex v_r gets color i only if $\alpha_i \geq 1$ and $\beta_i^{j(r)} = 0$. We therefore let $C(u) = \{i \in S : \alpha_i \geq 1$ and $\beta_i^{j(r)} = 0\}$ and put exactly $|C(u)|$ arcs entering into u, and give each such arc a *color* $i \in C(u)$ (exactly one color from $C(u)$ per arc). Each arc $(u', u) \in A$, with $u' \in N_{r-1}$ and $i \in C(u)$, will have then the following meaning: if the constrained sub-problem associated with u' has a solution, i.e., a coloring φ', then we can extend φ' into a solution φ to the constrained sub-problem associated with u by giving color i to vertex v_r.

We now give the formal definition of the set A. We start with the arcs from N_0 to N_1. Let $u = (1, \{\alpha_i\}_{i=1..s}, \{\beta_i^j\}_{i=1..s,j=1..k}) \in N_1$. There is an arc from $z_i, i \in S$, to u if and only if $i \in C(u)$, moreover, the color of this arc is i. We now deal with the arcs from N_{r-1} to N_r, with $2 \leq r \leq n$. Let $u = (r, \{\alpha_i\}_{i=1..s}, \{\beta_i^j\}_{i=1..s,j=1..k}) \in N_r$. As we discussed above, for each $i^* \in C(u)$, there will be an arc from a node $u_{i^*} \in N_{r-1}$ to u, with color i^*. Namely, $u_{i^*} = (r-1, \{\tilde{\alpha}_i\}_{i=1..s}, \{\tilde{\beta}_i^j\}_{i=1..s,j=1..k})$, where:

$$\tilde{\alpha}_i = \begin{cases} \alpha_i & i \neq i^* \\ \alpha_i - 1 & i = i^* \end{cases} \quad (1)$$

$$\tilde{\beta}_i^j = \begin{cases} \max\{|N(v_r, j)_<|, \beta_i^j\} & i = i^* \\ \max\{0, \beta_i^j - 1\} & i \neq i^*, j = j(r) \\ \beta_i^j & i \neq i^*, j \neq j(r) \end{cases} \quad (2)$$

Note that u_{i^*} is indeed a node of N_{r-1}, as the $(s(k+1)+1)$-tuple $(r-1, \{\tilde{\alpha}_i\}_{i=1..s}, \{\tilde{\beta}_i^j\}_{i=1..s,j=1..k})$ is such that $0 \leq \tilde{\alpha}_i \leq \alpha_i^*$, for each i, and $0 \leq \tilde{\beta}_i^j \leq \Delta(j)_<$, for each i, j (in fact, $\beta_i^j \leq \Delta(j)_<$, since u is a node of N_r).

Lemma 10. *G admits a capacitated s-coloring with capacities $\alpha_1^*, \ldots, \alpha_s^*$ if and only if there is a directed path from a node $z \in N_0$, to t in D. If such a path exists, then a capacitated s-coloring is that assigning each node v_r, $r = 1..n$, the color of the arc of the path entering into layer N_r.*

Proof. In order to prove the first part of the statement it is enough to show that property $(*)$ holds, for each node $u = (r, \{\alpha_i\}_{i=1..s}, \{\beta_i^j\}_{i=1..s,j=1..k}) \in N_r$, $1 \leq r \leq n$.

Since D is layered, u is reachable from a node $z \in N_0$ if either $r = 1$ and there is an arc from a node $z \in N_0$ to u, or $r > 1$ and there exists a node u' in N_{r-1}, reachable from a node $z \in N_0$ and such that (u', u) is an arc of D. We will therefore prove the property by induction on r, and we start with $r = 1$. First suppose that the arc (z_{i^*}, u) exists, for some $z_{i^*} \in N_0$. Then, by construction, $i^* \in C(u)$, i.e., $\alpha_{i^*} \geq 1$ and $\beta_{i^*}^{j(1)} = 0$; therefore, if we give v_1 color i^*, we obtain an s-coloring of $G[\{v_1\}]$ that is a solution for the constrained sub-problem associated with u. Vice versa, if there exists a coloring of $G[\{v_1\}]$ that is a solution to the constrained sub-problem associated with u, then the color i^* assigned to v_1 is such that $\alpha_{i^*} \geq 1$ and $\beta_{i^*}^{j(1)} = 0$, and so the arc (z_{i^*}, u) does exist.

Now, let $r > 1$. By inductive hypotheses, suppose that property $(*)$ holds true for nodes from the layer N_{r-1}. Suppose first that u is reachable from a node $z \in N_0$. Therefore there exists $u' = (r-1, \{\tilde{\alpha}_i\}_{i=1..s}, \{\tilde{\beta}_i^j\}_{i=1..s,j=1..k}) \in N_{r-1}$, such that: u' is reachable from a node $z \in N_0$; the arc (u', u) is an arc of A with color $i^* \in C(u)$; (1) and (2) hold true. By induction, $G[\{v_1, \ldots, v_{r-1}\}]$ admits a capacitated s-coloring φ', with capacities $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_s$, and the additional constraint that, for each $i = 1..s$ and $j = 1..k$, color i is not given to the last $\tilde{\beta}_i^j$ vertices in $V^j \cap \{v_1, \ldots, v_{r-1}\}$. We extend φ' to v_r by giving color i^* to v_r. It is easy to see that we then obtain a coloring φ that is a solution to the constrained sub-problem associated with u, in fact:

coloring constraints: for each class V^j, no vertex in $V^j \cap \{v_1, \ldots, v_{r-1}\}$ adjacent to v_r gets color i^*, as: φ' is not giving color i^* to the last $\tilde{\beta}^j_{i^*}$ vertices in $V^j \cap \{v_1, \ldots, v_{r-1}\}$; v_r is only adjacent to the last $|N(v_r, j)_<|$ vertices in $V^j \cap \{v_1, \ldots, v_{r-1}\}$ (see Fact 7); $|N(v_r, j)_<| \leq \tilde{\beta}^j_{i^*}$, by (2);

capacity constraints: φ satisfies capacity constraint because φ' satisfies capacity constraint for the constrained sub-problem associated with u' and (1) holds;

additional constraints: if we consider a class V^j that does not contain v_r, i.e., $j \neq j(r)$, and a color $i \neq i^*$, φ is not giving color i to the last β^j_i vertices in $V^j \cap \{v_1, \ldots, v_r\}$, as: φ' is not giving color i to the last $\tilde{\beta}^j_i$ vertices in $V^j \cap \{v_1, \ldots, v_{r-1}\}$; $V^j \cap \{v_1, \ldots, v_r\} = V^j \cap \{v_1, \ldots, v_{r-1}\}$; $\beta^j_i = \tilde{\beta}^j_i$ by (2);

if we consider the class $V^{j(r)}$ and a color $i \neq i^*$ such that $\beta^{j(r)}_i > 0$, φ is not giving color i to the last $\beta^{j(r)}_i$ vertices in $V^{j(r)} \cap \{v_1, \ldots, v_r\}$, as φ' is not giving color i to the last $\tilde{\beta}^{j(r)}_i$ vertices in $V^{j(r)} \cap \{v_1, \ldots, v_{r-1}\}$; v_r gets color $i^* \neq i$; $\tilde{\beta}^{j(r)}_i = \beta^{j(r)}_i - 1$ by (2);

if we consider a class V^j, $j \neq j(r)$, and the color i^*, φ is not giving color i^* to the last $\beta^j_{i^*}$ vertices in $V^j \cap \{v_1, \ldots, v_r\}$, as: φ' is not giving color i^* to the last $\tilde{\beta}^j_{i^*}$ vertices in $V^j \cap \{v_1, \ldots, v_{r-1}\}$; $V^j \cap \{v_1, \ldots, v_r\} = V^j \cap \{v_1, \ldots, v_{r-1}\}$; $\tilde{\beta}^j_{i^*} \geq \beta^j_{i^*}$, by (2);

if we consider the class $V^{j(r)}$ and the color i^*, there is no additional constraint, as $\beta^{j(r)}_{i^*} = 0$.

Conversely, let φ a capacitated s-coloring of $G[\{v_1, \ldots, v_r\}]$ that is a solution to the constrained sub-problem associated with u, and let $i^* = \varphi(v_r)$. So $\alpha_{i^*} \geq 1$ and $\beta^{j(r)}_{i^*} = 0$. Therefore, there exists a node $u' = (r-1, \{\tilde{\alpha}_i\}_{i=1..s}, \{\tilde{\beta}^j_i\}_{i=1..s,j=1..k}) \in N_{r-1}$ such that (u', u) is an arc of A with color i^*, and (1) - (2) hold. It is easy to see that, if we restrict φ to $\{v_1, \ldots, v_{r-1}\}$, then we get a capacitated s-coloring φ' that is a solution to the constrained sub-problem associated with u'. In fact, φ' is trivially an s-coloring of $G[\{v_1, \ldots, v_{r-1}\}]$; it satisfies capacity constraints by (1); finally it satisfies the additional constraints by construction (we skip the details, as they go along the same lines as above). By inductive hypotheses, therefore, u' is reachable from a node $z \in N_0$, and so is u.

Finally, if in D there exists a directed path from a node $z \in N_0$ to t, this path goes through one node of each layer. By construction, an arc

(u', u), with $u' \in N_{r-1}$ and $u \in N_r$, has color i if the following holds: if the constrained sub-problem associated with u' has a solution φ', then we can extend φ into a solution to the constrained sub-problem associated with u by giving color i to v_r. Therefore, if t is reachable from a node $z \in N_0$ via a path P, then a capacitated s-coloring of G is that assigning each node v_r, $r = 1..n$, the color of the arc of P entering into layer N_r. $\square$

Theorem 11. *Suppose that for a (k-thin) graph G with n vertices we are given an ordering and a partition of $V(G)$ into k classes that are consistent. Then, an instance (G, α^*, s) of the capacitated coloring problem can be solved in $O(n^{s+1}s^2k\prod_{j=1..k}\Delta(j)^s_<)$-time, that is $O(n^{ks+s+1}s^2k)$-time.*

Proof. By definition, for $r = 1..n-1$, $|N_r| = \prod_{i=1..s}(\alpha^*_i+1)\prod_{j=1..k}(\Delta(j)_<+1)^s$. Note that each node of D has at most s incoming arcs, and each arc can be built in $O((s(k+1))$-time. Therefore, D can be built in $O(n^{s+1}s^2k\prod_{j=1..k}\Delta(j)^s_<)$-time. Finally, since D is acyclic, the reachability problem on D can be solved in linear time in the number of arcs of D. So, the overall time complexity is $O(n^{s+1}s^2k\prod_{j=1..k}\Delta(j)^s_<)$, that is $O(n^{ks+s+1}s^2k)$. $\square$

Theorem 12. *On a co-comparability graph G with n vertices, an instance (G, α^*, s) of the capacitated coloring problem can be solved in $O(n^{s^2+s+1}s^3)$-time, that is polynomial when s is fixed.*

Proof. From Theorem 9 in time $O(n^3)$ we may either decide that the instance is infeasible, or provide an ordering and a k-partition of V, with $k \leq s$, which are consistent. In the latter case, from Theorem 11 the problem can be solved in time $O(n^{ks+s+1}s^2k) = O(n^{s^2+s+1}s^3)$-time. Since $s^2+s+1 \geq 3$, the statement follows. $\square$

Corollary 13. *An instance (π, h, s) of the Pickup and Delivery Tour Combination Problem with n items can be solved in $O(n^{s^2+s+1}s^3)$-time, and therefore in polynomial time when s is fixed.*

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