# Minimum weighted clique cover on claw-free perfect graphs

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**Abstract**

The first combinatorial algorithm for the minimum weighted clique cover (MWCC) in a claw-free perfect graph $G$ due to Hsu and Nemhauser [23] dates back to 1984. It is essentially a "dual" algorithm as it relies on any algorithm for the maximum weighted stable set (MWSS) problem in claw-free graphs and, taking into account the best known complexity for the latter problem, its complexity is $O(|V(G)|^5)$. More recently, Chudnovsky and Seymour [7] introduced a composition operation, strip-composition, in order to define their structural results for claw-free graphs; however, this composition operation is general and applies to non-claw-free graphs as well. In this paper, we show that a MWCC of a perfect strip-composed graph, with the basic graphs belonging to a class $\mathcal{G}$, can be found in polynomial time, provided that the MWCC problem can be solved on $\mathcal{G}$ in polynomial time. For the case of claw-free perfect strip-composed graphs, the algorithm can be tailored so that it never requires the computation of a MWSS on the strips and can be implemented as to run in $O(|V(G)|^3)$ time. Finally, building upon several results from the literature, we show how to deal with non-strip-composed claw-free perfect graphs, and therefore compute a MWCC in a general claw-free perfect graph in $O(|V(G)|^3)$ time.

**Keywords.** Claw-free graphs, line graphs, minimum weighted clique cover, perfect graphs, strip-composed graphs.

## 1 Introduction.

Given a graph $G$ and a non-negative weight function $w$ defined on the vertices of $G$, a *weighted clique cover* of $G$ is a collection of cliques, with a non-negative weight $y(K)$ assigned to each clique $K$ in the collection, such that, for each vertex $v$ of $G$, the sum of the weights of the cliques containing $v$ in the collection is at least $w(v)$. A *minimum weighted clique cover* of $G$ (MWCC) is a clique cover such that the sum of the weights of all the cliques in the collection is minimum. When all the vertex weights are 1, a (minimum) weighted clique cover is simply called a *(minimum) clique cover.*

Observe that a coloring of a graph is a clique cover of its complement. Since coloring in triangle-free graphs is NP-complete [26], it follows that MWCC is NP-complete already for graphs with stability number 2. However, things are easier for perfect graphs. First, it is known that for a perfect graph $G$, the weight $\tau_w(G)$ of a MWCC is the same as $\alpha_w(G)$, the weight of a *maximum weighted stable set* (MWSS) of $G$, that is, a set of pairwise non-adjacent vertices such that the sum of the weights of the vertices in the set is maximum [25]. Moreover, when $w$ is integral, there always exists an integer optimal solution to the MWCC problem, as it was originally shown by Fulkerson [12]. Finally, the weight of a MWCC in a perfect graph can be determined in polynomial time by using Lovász's theta function [16]. If one wants to compute also a MWCC of a perfect graph $G$ (and not only the number $\tau_w(G)$), a polynomial time algorithm proposed by Grötschel, Lovász and Schrijver in [17] can be used. This algorithm is not combinatorial and it uses the $\vartheta_w(G)$ function combined with other techniques; however, for some graph classes, among them particular classes of perfect graphs, there also exist polynomial time combinatorial algorithms, for the weighted or the unweighted (MCC) version [15, 18, 20, 22, 34].

This is the case, for instance, for claw-free perfect graphs, where combinatorial algorithms for both the unweighted and the weighted version have been proposed by Hsu and Nemhauser [21, 23]. To the best of our knowledge, the algorithm for the weighted case – in this paper, we mainly deal with this, as it is more general – was for more than 30 years the only available polynomial time combinatorial algorithm to solve MWCC in claw-free perfect graphs. (A graph is *claw-free* if none of its vertices has a stable set of size three in its neighborhood.) This algorithm is essentially a "dual" algorithm as it relies on any algorithm for the MWSS problem in claw-free graphs (we have, nowadays, several algorithms for this, see [10, 11, 28, 29, 30, 31, 32]), and, in fact, builds a MWCC by a clever use of linear programming complementarity slackness. The computational complexity of the algorithm by Hsu and Nemhauser, taking into account the best known complexity for MWSS, is $O(|V(G)|^5)$. (We point out that in the paper, we assume that the best known complexity for MWSS in a claw-free graph with $n$ vertices is $O(n^3)$ [11]. However, an $O(n^2 \log n)$ algorithm was recently announced by Nobili and Sassano [30].) Finally, it is worth noticing that the MCC and MWCC problems are NP-complete on general claw-free graphs and even for line graphs of triangle-free graphs: the latter fact follows from hardness of vertex cover in triangle-free graphs [14], since vertex cover of triangle-free graphs can be reduced to clique cover of their line graphs.

In the last years a lot of efforts have been devoted to a better understanding of the structure of perfect graphs and of other relevant classes of graphs. Claw-free graphs in particular have been investigated, with an outstanding series of papers by Chudnovsky and Seymour (for a survey see [7]). The results by Chudnovsky and Seymour show that claw-free graphs with stability number greater than three are either *fuzzy circular interval graphs* (a generalization of *proper circular arc graphs*, we do not give the definition, as it is not interesting for this paper) or *strip-composed*, i.e., they are suitable composition of some basic graphs (the formal definition is given in the next section). Exploiting this twofold structure of claw-free graphs has been the key for several developments for the MWSS problem [9, 31, 10, 11] and the dominating set problem [19]. In particular, in [31] it is shown that a MWSS of a (non-necessarily claw-free) strip-composed graph, with the basic graphs belonging to a class $\mathcal{G}$, can be found in polynomial time by solving a matching problem, provided that the MWSS problem can be solved on $\mathcal{G}$ in polynomial time. Building upon this result, new algorithms for the MWSS problem in claw-free graphs are given in [31] and [10, 11].

In this paper, we provide an analogous of the result in [31] for the MWCC problem. Namely, we show that a MWCC of a (non-necessarily claw-free) perfect strip-composed graph, with the basic graphs belonging to a class $\mathcal{G}$, can be found in polynomial time, provided that the MWCC problem can be solved on $\mathcal{G}$ in polynomial time. We point out that while the statement of this result goes along the same line as the result in [31], its proof requires new tools and techniques. We apply this result to strip-composed claw-free perfect graphs, and provide a $O(|V(G)|^3)$-time algorithm for the MWCC problem that, differently from the $O(|V(G)|^5)$-time dual algorithm by Hsu and Nemhauser, has both a primal and a primal-dual flavour: on each basic graph we directly compute a MWCC, while we use a primal-dual (matching) algorithm for the composition of basic graphs.

We finally deal with the MWCC for the class of claw-free perfect graphs that according to [31] are not strip decomposable. Exploiting some algorithmic results for clique cutset decomposition by Tarjan [33] and characterizations of claw-free perfect graphs without clique cutsets by Chvátal and Sbihi [8] and Maffray and Reed [27], we design an $O(|V(G)|^3)$-time combinatorial algorithm for this class as well. Altogether we show that for a claw-free perfect graph $G$ MWCC can be solved in time $O(|V(G)|^3)$.

The organization of the paper is as follows. We devote the remaining of this section to some basic definitions. In Section 2, we recall the MWSS algorithm for strip-composed graphs. In Section 3, we present in detail our main result, the MWCC algorithm for strip-composed *perfect* graphs (Theorem 2). We explain why the approach for MWSS needs to be adapted to deal with perfection, and we prove some necessary technical lemmas. In Section 4 we focus on claw-free perfect graphs. First, in Section 4.1 we show how to apply the MWCC algorithm of Section 3 to strip-composed claw-free perfect graphs; then in Section 4.2 we deal with {claw,net}-free perfect graphs that according to [31] are the non-strip decomposable claw-free (perfect) graphs: in Section 4.2.1 we deal with clique cutset decomposition trees, and tailor some existing results to the class of {claw,net}-free perfect graphs; in Section 4.2.2 and 4.2.3 we deal with MWCC in peculiar and elementary graphs, that are the atom graphs of the clique cutset decomposition tree.

*An extended abstract of this work was published in [1].*

### 1.1 Basic definitions.

We shall consider finite, simple, loopless, undirected graphs. When dealing with multigraphs, we will say so explicitly. Let $G$ be a graph. Denote by $V(G)$ its vertex set and by $E(G)$ its edge set.

The *neighborhood* of a vertex $v$ is the set $N(v)$ consisting of all the vertices which are adjacent to $v$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. A vertex $v$ of $G$ is *universal* if $N[v] = V(G)$. Two vertices $v$ and $w$ are *twins* if $N[v] = N[w]$.

For a subset $A$ of $V(G)$, let $N_A(v) = N(v) \cap A$. For subsets $A$ and $B$ of $V(G)$, let $N_A(B) = \bigcup_{v \in B} N_A(v)$.

For a subset $V' \subseteq V(G)$, the *j-th neighborhood* $N_j(V')$ is the set of vertices $u \in V(G)$ at distance $j$ from the set $V'$. When $V' = \{v\}$ we will write simply $N_j(v)$ and when $j = 1$ we will write just $N(V')$ (resp. $N(v)$).

We will denote by $G[V']$ the subgraph of $G$ induced by $V'$, and by $G \setminus V'$ the subgraph of $G$ induced by $V(G) \setminus V'$. Two sets $U, U' \subset V(G)$ are *complete* (*to each other*) if every vertex in $U$ is adjacent to all the vertices in $U'$. They are *anticomplete* (*to each other*) if no vertex of $U$ is adjacent to a vertex of $U'$.

A *claw* is a graph formed by a vertex with three neighbors of degree one. An *odd hole* is a chordless cycle of odd length at least 5. If $H$ is a graph, a graph $G$ is called *H-free* if no induced subgraph of $G$ is isomorphic to $H$.

The *intersection graph* of a family of sets $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$, two sets in $\mathcal{C}$ being adjacent if and only if they intersect. The *line graph* $L(G)$ of a graph or multigraph $G$ is the intersection graph of its edges. A graph $H$ is a line graph if there is a graph or multigraph $G$ such that $H = L(G)$ ($G$ is called a *root graph* of $H$). Given a multigraph, a *multistar* is the set of edges incident to one of its vertices, which is called the *center* of the multistar, while a *multitriangle* is the set of edges of the subgraph induced by three pairwise adjacent vertices. A *matching* is a set of pairwise non-incident edges of a graph (two edges are *incident* if they share a vertex). Note that the multistars and multitriangles of a graph $G$ correspond to the cliques of $L(G)$, while the matchings of $G$ correspond to the stable sets of $L(G)$. Note also that the neighborhood of a vertex in a line graph can be always covered by two cliques. A graph is *quasi-line* if the neighborhood of each vertex can be covered by two cliques. A quasi-line graph is, in particular, claw-free. Moreover, as observed by Hsu and Nemhauser in [23], a claw-free perfect graph is indeed quasi-line.

Finally, given two sets $A$ and $B$, we let $A \triangle B$ denote their symmetric difference.

## 2 The MWSS problem on strip-composed graphs.

Chudnovsky and Seymour [7] introduced a composition operation in order to define their structural results for claw-free graphs. This composition operation is general and applies to non-claw-free graphs as well.

A *strip* $H = (G, \mathcal{A})$ is a graph $G$ (not necessarily connected) with a multi-family $\mathcal{A}$ of either one or two designated non-empty cliques of $G$. The cliques in $\mathcal{A}$ are called the *extremities* of $H$, and $H$ is said a *1-strip* if $|\mathcal{A}| = 1$, and a *2-strip* if $|\mathcal{A}| = 2$. Let $\mathcal{G} = (G^1, \mathcal{A}^1), \dots, (G^k, \mathcal{A}^k)$ be a family of $k$ vertex disjoint strips, and let $\mathcal{P}$ be a partition of the multi-set of the cliques in $\mathcal{A}^1 \cup \ldots \cup \mathcal{A}^k$. The *composition* of the $k$ strips with respect to $\mathcal{P}$ is the graph $G$ that is obtained from the union of $G^1, \dots, G^k$, by making adjacent vertices of $A \in \mathcal{A}^i$ and $B \in \mathcal{A}^j$ ($i, j$ not necessarily different) if and only if $A$ and $B$ are in the same class of the partition $\mathcal{P}$. In this case we also say that $(\mathcal{G}, \mathcal{P})$, where $\mathcal{G} = \{(G^j, \mathcal{A}^j), j \in 1, \dots, k\}$, defines a *strip decomposition* of $G$.

We say that a graph $G$ is *strip-composed* if $G$ is a composition of some set of strips with respect to some partition $\mathcal{P}$. Each class of the partition of the extremities $\mathcal{P}$ defines a clique of the composed graph, that is called a *partition-clique*. We denote the extremities of the strip $H_i$ by $\mathcal{A}^i = \{A_1^i, A_2^i\}$ if $H_i$ is a 2-strip and by $\mathcal{A}^i = \{A_1^i\}$ if $H_i$ is a 1-strip. We often abuse notations, and when we refer to a vertex of a strip (or to a stable set of a strip etc.) we indeed consider a vertex (or a stable set etc.) of the graph in the strip.

The composition operation preserves some graph properties. Given a 2-strip $(G, \{A_1, A_2\})$, the graph $G_+$ is obtained from $G$ by adding two vertices $a_1, a_2$ such that $N(a_j) = A_j$, for $j = 1, 2$; for a 1-strip $(G, \{A_1\})$ the graph $G_+$ is obtained from $G$ by adding a vertex $a_1$ such that $N(a_1) = A_1$. A strip $(G, \mathcal{A})$ is said to be claw-free/quasi-line/line if the graph $G_+$ is claw-free/quasi-line/line. The composition of claw-free/quasi-line/line strips is a claw-free/quasi-line/line graph (see e.g. [10]).

Suppose we are given a strip-composed graph $G$ and its strip decomposition $(\mathcal{G}, \mathcal{P})$. In [31] it is shown how to exploit this decomposition in order to solve the MWSS on $G$.

**Theorem 1** *[31] Let $\mathcal{G}$ be the class of graphs which are the composition of strips $H_i = (G^i, \mathcal{A}^i)$ $i = 1, \dots, k$ with respect to a partition $\mathcal{P}$, such that each of the strips belongs to a class $\mathcal{C}$. Suppose also that there exists a function $p$ such that for each $G_i \in \mathcal{C}$ we can* *compute a* MWSS *in time $O(p(|V(G^i)|))$. Then the* MWSS *problem on $G \in \mathcal{G}$ can be solved in time $O(\sum_{i=1}^{k} p(|V(G^i)|) + match(|V(G)|))$, where $match(n)$ is the time required to solve the matching problem on a graph with $n$ vertices. If $p$ is a polynomial, then the* MWSS *can be solved on the class $\mathcal{G}$ in polynomial time.*

In order to prove their theorem [31], the authors replace every strip $H_i$ with a suitable simpler *gadget strip* $T_i$, that is a single vertex for each 1-strip and a triangle for each 2-strip (in this second case the extremities are two different edges of the triangle). Then they define a weight function on the vertices of those simpler strips; for every strip $H_i$ with extremities $A_1^i$ and $A_2^i$ this function depends on the values $\alpha_w(G^i)$, $\alpha_w(G^i \setminus A_1^i)$, $\alpha_w(G^i \setminus A_2^i)$ and $\alpha_w(G^i \setminus (A_1^i \cup A_2^i))$. Thus, if one can compute a MWSS of $G^i$ in polynomial time, then one can compute the weight function of the simpler strips in polynomial time.

They define a suitable partition $\tilde{\mathcal{P}}$ of the extremities of the gadget strips. In this way they obtain a graph $\tilde{G}$ which is the strip-composition of the strips $T_i$, $i = 1, \ldots, k$, with respect to the partition $\tilde{\mathcal{P}}$, and, since the strips are line strips, this graph is line. Moreover, from the construction of the simpler strips and of the weights, it is easy to translate a MWSS of $\tilde{G}$ into a MWSS of $G$. Finally, as $\tilde{G}$ is a line graph, they can find a MWSS of $\tilde{G}$ by building the root graph of $\tilde{G}$ and computing a maximum weighted matching in this graph.

## 3 The MWCC problem on strip-composed perfect graphs.

Suppose we are given a perfect graph $G$ that is the composition of strips $H_1 = (G^1, \mathcal{A}^1), \ldots, H_k = (G^k, \mathcal{A}^k)$ with respect to a partition $\mathcal{P}$, and a non-negative weight function $w$ on $V(G)$. In this section we will show how to exploit the decomposition in order to solve the MWCC on $G$. We will follow the approach outlined in the previous section for the MWSS and will compute a MWCC of $G$ in three steps:

Step 1. We replace each strip $H_i$ by a simple *gadget strip* $\tilde{H}_i = (\tilde{G}^i, \tilde{\mathcal{A}}^i)$, define a suitable weight function $\tilde{w}$ on the vertices of $\tilde{G}$ and compose the strips $\tilde{H}_i$ with respect to a suitable partition of the multi-set $\bigcup_{i=1..k} \tilde{\mathcal{A}}^i$ so as to obtain a graph $\tilde{G}$ that is line and perfect. Note that, in order to define $\tilde{w}$ and the right $\tilde{H}_i$, we need to find some MWCC (or MWSS) on $G^i$. The following will then hold: $\tau_w(G) = \tau_{\tilde{w}}(\tilde{G}) + \sum_{i=1}^{k} \delta_1^i$, where $\delta_1^i = \alpha_w(G^i \setminus A_1^i)$ for 1-strips and $\delta_1^i = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$ for 2-strips.

Step 2. Following [35], we find a MWCC $\tilde{y}$ of $\tilde{G}$, with respect to weights $\tilde{w}$, by running a primal-dual algorithm for the maximum weighted matching [13] on the root graph of $\tilde{G}$.

Step 3. We reconstruct a MWCC of $G$ with respect to the weight function $w$ from $\tilde{y}$, which is a MWCC of $\tilde{G}$ with respect to the weight function $\tilde{w}$. This step is indeed made of two substeps: first, we need to "translate" each (maximal) clique of $\tilde{G}$ into a clique of $G$ as to translate $\tilde{y}$ itself into a partial weighted clique cover $y$ of $G$ with value $\tau_{\tilde{w}}(\tilde{G})$; then, for each strip $H_i$, we have to "complete" $y$ with a suitable weighted clique cover of value $\delta_1^i$: that will be done by finding for $G^i$ a MWCC with respect to a suitable weight function $w^i$ defined on the vertices of $G^i$.

While this algorithm goes along the same lines of the one in [31] for the MWSS, things are more involved here and Steps 1 and 3 pose some challenges: **Step 1** we cannot use the gadget strips defined in the previous section for the MWSS, as the graph $\tilde{G}$ might be imperfect:

this will lead us to define four different new gadgets, with different parity properties, that are such that $\tilde{G}$ is odd hole free and line, thus perfect [35]; **Step 3** there is not always a direct correspondence between cliques of $\tilde{G}$ and cliques of $G$. Moreover, for some 2-strips $H_i = (G^i, \mathcal{A}^i)$, in order to complete $y$ we will need to compute a MWCC of some auxiliary graphs associated to the strip, rather than to the strip itself: the graph $G^i_\bullet$ that is obtained from $G^i$ by adding a new vertex $x$ complete to both $A^i_1$ and $A^i_2$ and the graph $G^i_=$ that is the graph obtained from $G^i$ by making $A^i_1$ complete to $A^i_2$.

In spite of these difficulties, we will be able to prove the following:

**Theorem 2** *Let $\mathcal{G}$ be the class of perfect graphs which are the composition of strips $H_i = (G^i, \mathcal{A}^i)$ $i = 1, \dots, k$ with respect to a partition $\mathcal{P}$, such that each of the strips belongs to a class $\mathcal{C}$. Suppose also that there exists a function $p$ such that for each strip $H_i = (G^i, \mathcal{A}^i)$ in $\mathcal{C}$ we can compute in time $O(p(|V(G^i)|))$ a MWCC of $G^i$, a MWCC of $G^i_\bullet$ and a MWCC of $G^i_=$. Then the MWCC problem on $G \in \mathcal{G}$ can be solved in time $O(\sum_{i=1}^k p(|V(G^i)|) + match(|V(G)|))$, where $match(n)$ is the time required to solve the matching problem on a graph with $n$ vertices. If $p$ is a polynomial, then the MWCC can be solved on the class $\mathcal{G}$ in polynomial time.*

We later provide a slightly technical improvement for Theorem 2, as we will characterize in which cases we indeed need to compute MWCC-s for $G^i_\bullet$ and $G^i_=$. We devote the remaining of this section to provide more details and a proof.

### 3.1 The gadget strips.

We first deal with the gadget strips (that in this section we simply call gadgets) that will compose the graph $\tilde{G}$ and establish the relation between $\tau_w(G)$ and $\tau_{\tilde{w}}(\tilde{G})$. We make a heavy use of duality between the MWCC and the MWSS problem: the fact that for every induced subgraph $J$ of $G$, $\alpha_w(J) = \tau_w(J)$, is due to the perfection of $G$. We use this relation to easily prove the correctness of the weight function defined on the vertices of each gadget.

To design the gadgets, we delve into three cases: $(i)$ $H_i = (G^i, \mathcal{A}^i)$ is a 1-strip; $(ii)$ $H_i = (G^i, \mathcal{A}^i)$ is a 2-strip with the extremities in the same class of the partition $\mathcal{P}$; $(iii)$ $H_i = (G^i, \mathcal{A}^i)$ is a 2-strip with the extremities in different classes of the partition.

$(i) - (ii)$ In these cases, the gadget will be a single vertex (see Figure 1). In particular we define the 1-strip $\tilde{H}^0_i = (T^i_0, \tilde{\mathcal{A}}^i_0)$, where the graph $T^i_0$ consists of a single vertex $c^i$, and $\tilde{\mathcal{A}}^i_0 = \{\{c^i\}\}$. For case $(i)$, we let $\delta^i_1 = \alpha_w(G^i \setminus A^i_1)$ and define $\tilde{w}(c^i) = \alpha_w(G^i) - \delta^i_1$. For case $(ii)$, we let $\delta^i_1 = \alpha_w(G^i \setminus (A^i_1 \cup A^i_2))$ and define $\tilde{w}(c^i) = \max\{\alpha_w(G^i \setminus A^i_1), \alpha_w(G^i \setminus A^i_2), \alpha_w(G^i \setminus (A^i_1 \triangle A^i_2))\} - \delta^i_1$. Finally, when we replace $H_i$ by $\tilde{H}^0_i$, we define a new partition $\mathcal{P}' := (\mathcal{P} \setminus \{P\}) \cup \{(P \setminus \mathcal{A}^i) \cup \tilde{\mathcal{A}}^i_0\}$, where $P \in \mathcal{P}$ was the set containing $\mathcal{A}^i$. We get the following lemma, whose proof is postponed to the Appendix, as it goes along the same lines as the proof of Theorem 1 in [31]:

**Lemma 3** *Let $G$ be the composition of strips $H_1 = (G^1, \mathcal{A}^1), \dots, H_k = (G^k, \mathcal{A}^k)$ with respect to a partition $\mathcal{P}$, and let $w$ be a non-negative weight function defined on the vertices of $G$. Suppose that $H_1$ is either a 1-strip or a 2-strip with the extremities in the same class of the partition $\mathcal{P}$. Let $G'$ be the composition of strips $\tilde{H}^0_1 = (T^1_0, \mathcal{A}^1_0), H_2 = (G^2, \mathcal{A}^2), \dots, H_k = (G^k, \mathcal{A}^k)$ with respect to the partition $\mathcal{P}'$. Let $w'$ be the weight function defined on the vertices of $G'$ as $w'(v) = w(v)$ for $v \in \bigcup_{i=2..k} V(G^i)$, and $w'(c^1) = \tilde{w}(c^1)$. Then $\alpha_w(G) = \alpha_{w'}(G') + \delta^1_1$.*

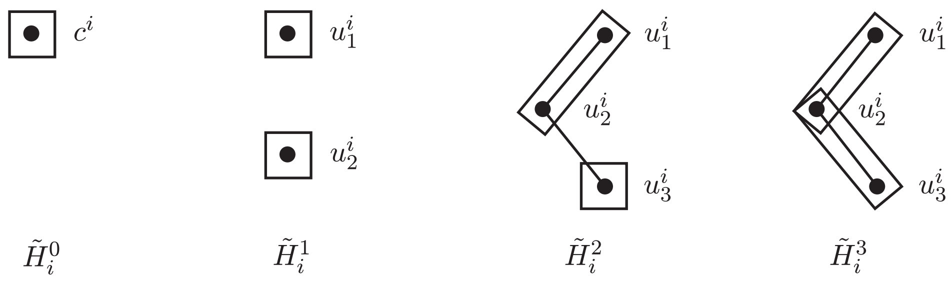


Figure 1: The strips $\tilde{H}_i^0, \tilde{H}_i^1, \tilde{H}_i^2, \tilde{H}_i^3$, possibly associated with the strip $H_i$.

*(iii)* Let us now consider a 2-strip $H_i = (G^i, \mathcal{A}^i)$ with the extremities in different classes of the partition $\mathcal{P}$. We want again to introduce a gadget $\tilde{H}_i = (\tilde{G}^i, \tilde{\mathcal{A}}^i)$ and a weight function $\tilde{w}$ on the vertices of $\tilde{G}^i$ in such a way that, when we replace $H_i$ by $\tilde{H}_i$ and define a new suitable partition, the difference between the weights of the MWSS of the original graph and the MWSS of the new graph is $\delta_1^i$, where this time $\delta_1^i = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$.

This is satisfied by the gadget strip defined in [31] for replacing 2-strips, but for that strip, a triangle with vertices $\{a, b, c\}$ and extremities $\{a, b\}$ and $\{b, c\}$, there is both an even and an odd length path between vertices of the extremities (respectively, the path of length 0 made of vertex $b$, and the path of length 1 $a - c$). It follows that by using that gadget we would easily introduce odd holes in the resulting graph $\tilde{G}$ and therefore lose perfection! This is indeed the first technical challenge we have to face for implementing the algorithm we sketched above: define new suitable gadgets that take into account the parity of the strips. Before doing that, we need some additional definitions.

Let $U, W \subseteq V(G)$. We call a path $P = v_1, \ldots, v_k$ ($k \geq 1$) a *$U$–$W$ path* if $P$ is chordless, $v_1 \in U$, $v_k \in W$, and $v_i \notin U \cup W$ for $2 \leq i \leq k-1$. A 2-strip $H_i = (G^i, \mathcal{A}^1 = \{A_1^i, A_2^i\})$ will be called *non-connected* if there is no $A_1^i$–$A_2^i$ path, and *connected* otherwise. We say that a connected 2-strip $H_i$ is *even* (resp. *odd*) if every $A_1^i$–$A_2^i$ path has even (resp. odd) length. If a connected 2-strip has both even and odd length $A_1^i$–$A_2^i$ paths, then we say that $H_i$ is an *even-odd* strip. We call an odd or even-odd strip $H_i$ *odd-short* if every odd $A_1^i$–$A_2^i$ path has length one, and we call an even or even-odd strip $H_i$ *even-short* if every even $A_1^i$–$A_2^i$ path has length zero. [3], $H_i$ is an odd strip if and only if $A_1^i$ and $A_2^i$ are an odd pair of cliques in $G^i$.)

We introduce three different 2-strip gadgets: a non-connected strip, an odd strip that is also odd-short, and an even strip that is also even-short (see Figure 1):

(a) $\tilde{H}_i^1 = (T_1^i, \tilde{\mathcal{A}}_1^i)$ such that $V(T_1^i) = \{u_1^i, u_2^i\}$, $E(T_1^i) = \emptyset$, $\tilde{\mathcal{A}}_1^i = \{\tilde{A}_1^i, \tilde{A}_2^i\}$ and $\tilde{A}_1^i = \{u_1^i\}$, $\tilde{A}_2^i = \{u_2^i\}$. The new weight function $\tilde{w}$ gives the following weights to the vertices of $T_1^i$: $\tilde{w}(u_1^i) = \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, $\tilde{w}(u_2^i) = \alpha_w(G^i \setminus A_1^i) - \delta_1^i$.

(b) $\tilde{H}_i^2 = (T_2^i, \tilde{\mathcal{A}}_2^i)$ such that $V(T_2^i) = \{u_1^i, u_2^i, u_3^i\}$, $E(T_2^i) = \{u_1^i u_2^i, u_2^i u_3^i\}$, $\tilde{\mathcal{A}}_2^i = \{\tilde{A}_1^i, \tilde{A}_2^i\}$ and $\tilde{A}_1^i = \{u_1^i, u_2^i\}$, $\tilde{A}_2^i = \{u_3^i\}$. The new weight function $\tilde{w}$ gives the following weights to the vertices of $T_2^i$: $\tilde{w}(u_1^i) = \alpha_w(G^i) - \alpha_w(G^i \setminus A_1^i)$, $\tilde{w}(u_2^i) = \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, $\tilde{w}(u_3^i) = \alpha_w(G^i \setminus A_1^i) - \delta_1^i$.

(c) $\tilde{H}_i^3 = (T_3^i, \tilde{\mathcal{A}}_3^i)$ such that $V(T_3^i) = \{u_1^i, u_2^i, u_3^i\}$, $E(T_3^i) = \{u_1^i u_2^i, u_2^i u_3^i\}$, $\tilde{\mathcal{A}}_3^i = \{\tilde{A}_1^i, \tilde{A}_2^i\}$ and $\tilde{A}_1^i = \{u_1^i, u_2^i\}$, $\tilde{A}_2^i = \{u_2^i, u_3^i\}$. The new weight function $\tilde{w}$ gives the following weights to the vertices of $T_3^i$: $\tilde{w}(u_1^i) = \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, $\tilde{w}(u_2^i) = \alpha_w(G^i) - \delta_1^i$, $\tilde{w}(u_3^i) = \alpha_w(G^i \setminus A_1^i) - \delta_1^i$.

We are now ready to state which gadget strip among $\tilde{H}_i^1$, $\tilde{H}_i^2$ and $\tilde{H}_i^3$ should be used to replace $H_i$. Note that we will *not* evaluate the parity of $H_i$, but rather investigate the sign of a certain inequality, that is indeed related to the parity of the strips (see Lemma 6):

**Scheme 4** *The strip $H_i$ will be replaced by:*

*(c1)* $\tilde{H}_i^1$, *if* $\alpha_w(G^i \setminus A_1^i) + \alpha_w(G^i \setminus A_2^i) = \alpha_w(G^i) + \delta_1^i$;

*(c2)* $\tilde{H}_i^2$, *if* $\alpha_w(G^i \setminus A_1^i) + \alpha_w(G^i \setminus A_2^i) > \alpha_w(G^i) + \delta_1^i$;

*(c3)* $\tilde{H}_i^3$, *if* $\alpha_w(G^i \setminus A_1^i) + \alpha_w(G^i \setminus A_2^i) < \alpha_w(G^i) + \delta_1^i$.

We are now ready to state a lemma that is the analogous of Lemma 3 for 2-strips, however, in this case, $\delta_1^i = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$. Note that, whichever is the gadget strip $\tilde{H}_i^j$ we use to replace $H_i$, we define a new partition $\mathcal{P}' := \mathcal{P} \setminus \{P_1, P_2\} \cup \{(P_1 \setminus \{A_1^i\}) \cup \{\tilde{A}_1^i\}, (P_2 \setminus \{A_2^i\}) \cup \{\tilde{A}_2^i\}\}$, where $P_1, P_2 \in \mathcal{P} : A_1^i \in P_1, A_2^i \in P_2$. The proof is again postponed to the Appendix:

**Lemma 5** *Let $G$ be the composition of strips $H_1 = (G^1, \mathcal{A}^1), \ldots, H_k = (G^k, \mathcal{A}^k)$ with respect to a partition $\mathcal{P}$, and let $w$ be a non-negative weight function defined on the vertices of $G$. Suppose that $H_1$ is a 2-strip with the extremities in different classes of the partition $\mathcal{P}$ and replace it by the strip $\tilde{H}_1^j$, for some $j \in \{1, 2, 3\}$, as discussed above. Let $G'$ be the composition of strips $\tilde{H}_1^j = (T_j^1, \mathcal{A}_j^1), H_2 = (G^2, \mathcal{A}^2), \ldots, H_k = (G^k, \mathcal{A}^k)$ with respect to the partition $\mathcal{P}'$. Let $w'$ be the weight function defined on the vertices of $G'$ as $w'(v) = w(v)$ for $v \in \bigcup_{i=2..k} V(G^i)$, and $w'(v) = \tilde{w}(v)$ for $v \in V(T_j^1)$. Then $\alpha_w(G) = \alpha_{w'}(G') + \delta_1^1$.*

We now deal with perfection. We want also $G'$ to be odd hole free, so that, when we have eventually replaced each strip, the resulting graph $\tilde{G}$ is odd hole free and line, thus perfect [35]. Note that in order to show that $G'$ is odd hole free, it is enough to show that $\tilde{H}_i^j$ has no even paths if $H_i$ has none, and $\tilde{H}_i^j$ has no odd paths if $H_i$ has none. However, the only gadget strip with even paths is $\tilde{H}_i^3$ while the only gadget strip with odd paths is $\tilde{H}_i^2$: therefore we need to show that we use $\tilde{H}_i^3$ only if the strip is even or even-odd and that we use $\tilde{H}_i^2$ only if the strip is odd or even-odd. Following Scheme 4, it will be enough to show that we do not use $\tilde{H}_i^3$ when $H_i$ is odd or non-connected and analogously we do not use $\tilde{H}_i^2$ when $H_i$ is even or non-connected. That immediately follows from the next lemma, which shows that the fact that the relation $\alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1 \setminus A_2^1) \gtreqless \alpha_w(G^1) + \delta_1^1$ is satisfied with $=$, $>$, or $<$ is strictly related to connection and parity of $H_1 = (G^1, \mathcal{A}^1)$.

**Lemma 6** *Let $H_1 = (G^1, \mathcal{A}^1)$ be a 2-strip. Then:*
*(a) if $H_1$ is non-connected, then $\alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1 \setminus A_2^1) = \alpha_w(G^1) + \alpha_w(G^1 \setminus (A_1^1 \cup A_2^1))$;*
*(b) if $H_1$ is odd and $G^1$ perfect, then $\alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1 \setminus A_2^1) \geq \alpha_w(G^1) + \alpha_w(G^1 \setminus (A_1^1 \cup A_2^1))$;*
*(c) if $H_1$ is even and $G^1$ perfect, then $\alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1 \setminus A_2^1) \leq \alpha_w(G^1) + \alpha_w(G^1 \setminus (A_1^1 \cup A_2^1))$.*

*Proof.* The proof requires a crucial result on the structure of perfect graphs. An *odd antihole* is the complement of an odd hole. The Strong Perfect Graph Theorem, claiming that a graph is perfect if and only if it contains neither odd holes nor odd antiholes as induced subgraphs, was conjectured by Berge in the sixties and proved forty years later by Chudnovsky, Robertson, Seymour, and Thomas [5].

In the proof, we let $\delta_1^1 = \alpha_w(G^1 \setminus (A_1^1 \cup A_2^1))$. (a) Let $G_1^1$ be the connected component of $G^1$ that contains $A_1^1$, and let $G_2^1 = G^1 \setminus V(G_1^1)$. Since $H_1$ is a non-connected strip, then $A_2^1$ is contained in $G_2^1$. The following equalities are straightforward, and imply the lemma.

$$\begin{aligned}
\alpha_w(G^1) &= \alpha_w(G_1^1) + \alpha_w(G_2^1) \\
\alpha_w(G^1 \setminus A_2^1) &= \alpha_w(G_1^1) + \alpha_w(G_2^1 \setminus A_2^1) \\
\alpha_w(G^1 \setminus A_1^1) &= \alpha_w(G_1^1 \setminus A_1^1) + \alpha_w(G_2^1) \\
\delta_1^1 &= \alpha_w(G_1^1 \setminus A_1^1) + \alpha_w(G_2^1 \setminus A_2^1)
\end{aligned}$$

(b) Let $G^*$ be the graph obtained in the following way: add to $G^1$ a vertex $v_1$ complete to $A_1^1$, a vertex $v_2$ complete to $A_2^1$, with $v_1$ adjacent to $v_2$. As $H_1$ is odd, $A_1^1 \cap A_2^1 = \emptyset$. Besides, as $G^1$ is perfect, $G^*$ is perfect. Indeed, no odd hole or odd antihole contains a *simplicial* vertex (a vertex whose neighborhood is a clique). So, any possible odd hole or antihole in $G^*$ should contain both $v_1$ and $v_2$. On the one hand, if there is an odd hole containing $v_1$ and $v_2$, then there is an even $A_1^1$–$A_2^1$ path, a contradiction. On the other hand, any pair of vertices in an odd antihole of length at least 7 has a common neighbor, a contradiction because $A_1^1 \cap A_2^1 = \emptyset$. We want to extend $w$ to $v_1$ and $v_2$. In order to do that, we choose $a, b \geq 0$ and such that $a + \alpha_w(G^1 \setminus A_1^1) = b + \alpha_w(G^1 \setminus A_2^1) > \alpha_w(G^1)$: that is always possible. Then we let $w(v_1) = a$ and $w(v_2) = b$. A stable set in $G^*$ can either take $v_1$ and then no vertex of $A_1^1$, or it can take $v_2$ and then no vertex of $A_2^1$ or it can miss both $v_1$ and $v_2$. Then for our choice of the weights of $v_1$ and $v_2$ we have that $\alpha_w(G^*) = a + \alpha_w(G^1 \setminus A_1^1) = b + \alpha_w(G^1 \setminus A_2^1)$.

Let $y$ be a MWCC of $G^*$ with respect to $w$ and denote with $\tau_w(G^*)$ its value; let $h_1$ be the value given by $y$ to the clique $\{v_1\} \cup A_1^1$, $h_3$ be the value given by $y$ to the clique $\{v_2\} \cup A_2^1$, and let $h_2$ be the value given by $y$ to the clique $\{v_1, v_2\}$. Now we define a new weight function $w'$ on $V(G^1)$: $w'(v) = w(v)$ for every $v \in V(G^1) \setminus (A_1^1 \cup A_2^1)$, $w'(v) = w(v) - h_1$ for every $v \in A_1^1$, $w'(v) = w(v) - h_3$ for every $v \in A_2^1$. Denote with $\tau_{w'}(G^1)$ the value of a MWCC of $G^1$ with respect to $w'$, then $\tau_w(G^*) = h_1 + h_2 + h_3 + \tau_{w'}(G^1)$ by the optimality of $y$ for $G^*$ and the definition of $h_1, h_2, h_3$.

As $G^*$ is perfect we know that the maximum weight stable set problem and the minimum weighted clique cover on $G^*$ are dual problems and so every vertex $v$ belonging to a MWSS of $G^*$ is covered exactly by a MWCC of $G^*$, that is $\sum_{C \in \mathcal{K}(G^*): v \in K} y(K) = w(v)$, where $\mathcal{K}(G^*)$ is the set of maximal cliques of $G^*$. In particular, for our choice of $a$ and $b$, both $v_1$ and $v_2$ belong to MWSS of $G^*$, so we have that $h_1 + h_2 = a$ and $h_2 + h_3 = b$. Moreover, again by duality, $\alpha_w(G^*) = \tau_w(G^*)$, and we obtain $h_1 + h_2 + h_3 + \tau_{w'}(G^1) = a + \alpha_w(G^1 \setminus A_1^1)$, that is $h_3 + \tau_{w'}(G^1) = \alpha_w(G^1 \setminus A_1^1)$, and $h_1 + h_2 + h_3 + \tau_{w'}(G^1) = b + \alpha_w(G^1 \setminus A_2^1)$, that is $h_1 + \tau_{w'}(G^1) = \alpha_w(G^1 \setminus A_2^1)$. But again by duality and by the perfection of $G^1$ we can rewrite those two equations as (i) $h_3 + \alpha_{w'}(G^1) = \alpha_w(G^1 \setminus A_1^1)$ and (ii) $h_1 + \alpha_{w'}(G^1) = \alpha_w(G^1 \setminus A_2^1)$.

As $A_1^1$ and $A_2^1$ are cliques we can easily deduce the inequality $\alpha_w(G^1) \leq \alpha_{w'}(G^1) + h_1 + h_3$ and by definition of the weight function $w'$, it follows that $\delta_1^1 \leq \alpha_{w'}(G^1)$; summing up these inequalities we obtain $\alpha_w(G^1) + \delta_1^1 \leq 2\alpha_{w'}(G^1) + h_1 + h_3$, then using (i) and (ii) $\alpha_w(G^1) + \delta_1^1 \leq \alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1 \setminus A_2^1)$.

(c) We build the following auxiliary strip $H^* = (G^*, \mathcal{A}^*)$: $G^*$ is obtained from $G^1$ by adding a vertex $v$ complete to $A_1^1$, and $\mathcal{A}^* = \{\{v\}, A_2^1\}$. We observe that by construction and the hypothesis on $H_1$, $H^*$ is an odd strip and $G^*$ is perfect (adding a simplicial vertex preserves perfection). We extend the weight function of $G^1$ to $v$, putting $w(v) = a$, where $a > \alpha_w(G^1)$.

From the choice of $a$ we have the following equalities:

$$\begin{array}{rcl} \alpha_w(G^* \setminus \{v\}) & = & \alpha_w(G^1) \\ \alpha_w(G^* \setminus (\{v\} \cup A_2^1)) & = & \alpha_w(G^1 \setminus A_2^1) \\ \alpha_w(G^*) \;=\; \max\{a + \alpha_w(G^1 \setminus A_1^1), \alpha_w(G^1)\} & = & a + \alpha_w(G^1 \setminus A_1^1) \\ \alpha_w(G^* \setminus A_2^1) \;=\; \max\{a + \delta_1^1, \alpha_w(G^1 \setminus A_2^1)\} & = & a + \delta_1^1 \end{array}$$

By Lemma 6 (b), the following inequality holds $\alpha_w(G^* \setminus \{v\}) + \alpha_w(G^* \setminus A_2^1) \geq \alpha_w(G^* \setminus (\{v\} \cup A_2^1)) + \alpha_w(G^*)$, that is, $\alpha_w(G^1) + a + \delta_1^1 \geq \alpha_w(G^1 \setminus A_2^1) + a + \alpha_w(G^1 \setminus A_1^1)$ and therefore $\alpha_w(G^1) + \delta_1^1 \geq \alpha_w(G^1 \setminus A_2^1) + \alpha_w(G^1 \setminus A_1^1)$. □

Lemma 6 was present in the preliminary version of this paper [1], and a very similar result was independently proved for 2-joins by Trotignon and Vušković in [34]. 2-Joins and compositions of strips are very close concepts but neither are exactly the same nor one generalizes the other. Thus, for the sake of completeness, we keep the statement in terms of strips and its proof.

### 3.2 Finding a MWCC in $\tilde{G}$.

Suppose that $G$ is the composition of the strips $H_1, H_2, \ldots, H_k$ with respect to a partition $\mathcal{P}$ and suppose that $G$ is perfect. We also assume that we know for each strip the values of maximum weighted stable sets $\alpha_w(G^i)$, $\alpha_w(G^i \setminus A_2^i)$, $\alpha_w(G^i \setminus A_1^i)$ and $\alpha_w(G^i \setminus (A_1^i \cup A_2^i))$ or, analogously (given perfection), the values of the minimum weighted clique covers $\tau_w(G^i)$, $\tau_w(G^i \setminus A_2^i)$, $\tau_w(G^i \setminus A_1^i)$ and $\tau_w(G^i \setminus (A_1^i \cup A_2^i))$. We replace each strip $H_i$ that is either a 1-strip or a 2-strip with the extremities in the same class of the partition $\mathcal{P}$ by $\tilde{H}_i^0$, and the other 2-strip in agreement with Scheme 4 by the suitable gadget among $\tilde{H}_i^1, \tilde{H}_i^2$ and $\tilde{H}_i^3$. We end up with a graph $\tilde{G}$, that is the composition of $\tilde{H}_1^{j_1}, \tilde{H}_2^{j_2}, \ldots, \tilde{H}_k^{j_k}$ with respect to a suitable partition $\mathcal{P}'$, and with a weight function $\tilde{w}$ on the vertices of $\tilde{G}$ that is such that $\alpha_w(G) = \alpha_{\tilde{w}}(\tilde{G}) + \sum_{i=1}^k \delta_1^i$. Moreover, as we now shortly discuss, $\tilde{G}$ is perfect. Recall in fact that $G$ is perfect, and therefore odd hole free. Following Lemma 6, we replaced each 2-strip by a 2-strip (gadget) with the same parity and connection properties: it follows that $\tilde{G}$ is odd hole free too. Moreover, since $\tilde{H}_i^0, \tilde{H}_i^1, \tilde{H}_i^2, \tilde{H}_i^3$ are line strips, also $\tilde{G}$ is line; since a graph that is odd hole free and line is perfect [35], it follows that $\tilde{G}$ is perfect. Finally, since both $G$ and $\tilde{G}$ are perfect, it follows by duality that $\tau_w(G) = \tau_{\tilde{w}}(\tilde{G}) + \sum_{i=1}^k \delta_1^i$.

Let $H$ be a multigraph that is a root of $\tilde{G}$. Following [35], we may find a MWCC $\tilde{y}$ of $\tilde{G}$, with respect to the weight $\tilde{w}$, by running a primal-dual algorithm for the maximum weighted matching [13] on $H$: this is because each maximal clique of $\tilde{G}$ corresponds to either a multistar of $H$ or to a multitriangle of $H$. While every multitriangle of $H$ corresponds to a maximal clique of $\tilde{G}$, a multistar of $H$ corresponds to a maximal clique of $\tilde{G}$ if and only if it is not contained in a multitriangle. Without loss of generality, we my also assume that the weight given by $\tilde{y}$ is non-zero only for maximal cliques of $\tilde{G}$.

### 3.3 From a MWCC of $\tilde{G}$ to a MWCC of $G$.

We are left with "translating" $\tilde{y}$, which is a MWCC of $\tilde{G}$, with respect to weights $\tilde{w}$, into a MWCC of $G$, with respect to weights $w$. As we already discussed, this step is indeed made of two substeps: first, we need to translate each (maximal) clique of $\tilde{G}$ into a clique of $G$ as to translate $\tilde{y}$ itself into a partial weighted clique cover $y$ of $G$ with value $\tau_{\tilde{w}}(\tilde{G})$; then, for each strip $H_i$, we have to complete $y$ with a suitable weighted clique cover of value $\delta_1^i$.

However, already for the first substep, there is a catch: unfortunately there are some maximal cliques of $\tilde{G}$ that do not correspond to any clique of $G$. Dealing with this problem, which once again does not show up for the MWSS in [31], will require some work. We start with detailing the structure of $\tilde{G}$ and $H$.

### 3.3.1 The structure of $\tilde{G}$ and $H$.

We first show how to build $H$. Krausz [24] proved the following:

**Lemma 7** *[24] A graph $J(W, F)$ is the line graph of a multigraph if and only if there exists a family of cliques $\mathcal{Q}$ such that every edge in $F$ is covered by a clique from the family $\mathcal{Q}$, and moreover every vertex in $W$ belongs to exactly two cliques from the family $\mathcal{Q}$.*

In fact, as soon as we are given a family $\mathcal{Q}$ satisfying Lemma 7 with respect to $\tilde{G}$, we may build $H$ as follows: each clique $K \in \mathcal{Q}$ corresponds to a vertex $v_K$ of $H$, and two vertices $v_{K_1}$ and $v_{K_2}$ of $H$ are connected by $|K_1 \cap K_2|$ (parallel) edges. In order to build $\mathcal{Q}$, and therefore $H$, we start from the set of partition cliques defined by $\mathcal{P}'$. Note that each vertex of $\tilde{G}$ belongs to exactly one partition clique, but for each vertex $u_2^i$ from each strip $\tilde{H}_i^3$, as such a vertex belongs to exactly two partition cliques. Also note that each edge of $\tilde{G}$ is covered by a partition clique, but for each edge $u_2^i u_3^i$ from each strip $\tilde{H}_i^2$. Therefore, in order to "complete" $\mathcal{Q}$, we consider, besides the partition cliques, the following set of *completion* cliques of $\tilde{G}$: a clique $\{v\}$ for each vertex $v$ from each strip $\tilde{H}_i^0$ or $\tilde{H}_i^1$; a clique $\{v\}$ for each vertex $v \equiv u_1^i$ from each strip $\tilde{H}_i^2$; a clique $\{v\}$ for each vertex $v \in \{u_1^i, u_3^i\}$ from each strip $\tilde{H}_i^3$; a clique $\{u_2^i, u_3^i\}$ from each strip $\tilde{H}_i^2$. It is easy to see that the union of the partition and the completion cliques satisfies Lemma 7. The next remark summarizes the structure of $H$.

**Remark 1** *Suppose that $\mathcal{P} = \{P_1, \ldots, P_r\}$. Then $H$ is composed by:*

- *a set of vertices $\{x_1, \ldots, x_r\}$, each $x_i$ corresponding to the class $P_i$ of $\mathcal{P}$;*
- *a vertex $z_j^i$ and an edge $z_j^i x_j$, for each strip $H_i$ such that we use $\tilde{H}_i^0$ in the composition and such that $A_1^i \in P_j$ (the edge corresponds to the vertex $c^i$ of $T_0^i$);*
- *vertices $z_j^i$ and $z_\ell^i$ and edges $z_j^i x_j$ and $z_\ell^i x_\ell$, for each strip $H_i$ such that we use $\tilde{H}_i^1$ in the composition and such that $A_1^i \in P_j$ and $A_2^i \in P_\ell$ (the edges $z_j^i x_j$ and $z_\ell^i x_\ell$ correspond to the vertices $u_1^i$ and $u_2^i$ of $T_1^i$, respectively);*
- *a vertex $z_j^i$ and an edge $z_j^i x_j$, for each strip $H_i$ such that we use $\tilde{H}_i^2$ in the composition and such that $A_1^i \in P_j$ (the edge corresponds to the vertex $u_1^i$ of $T_2^i$);*
- *a vertex $y_{j,\ell}^i$ and edges $y_{j,\ell}^i x_j$ and $y_{j,\ell}^i x_\ell$, for each strip $H_i$ such that we use $\tilde{H}_i^2$ in the composition and such that $A_1^i \in P_j$ and $A_2^i \in P_\ell$ (the edges $y_{j,\ell}^i x_j$ and $y_{j,\ell}^i x_\ell$ correspond to the vertices $u_2^i$ and $u_3^i$ of $T_2^i$, respectively);*
- *vertices $z_j^i$ and $z_\ell^i$ and edges $z_j^i x_j$ and $z_\ell^i x_\ell$, for each strip $H_i$ such that we use $\tilde{H}_i^3$ in the composition and such that $A_1^i \in P_j$ and $A_2^i \in P_\ell$ (the edges $z_j^i x_j$ and $z_\ell^i x_\ell$ correspond to the vertices $u_1^i$ and $u_3^i$ of $T_3^i$, respectively);*

- *an edge $x_j x_\ell$, for each strip $H_i$ such that we use $\tilde{H}_i^3$ in the composition, where $A_1^i \in P_j$ and $A_2^i \in P_\ell$ (this edge corresponds to the vertex $u_2^i$ of $T_3^i$).*

### 3.3.2 The maximal cliques of $\tilde{G}$.

We now analyze the maximal cliques of $\tilde{G}$. It is easy to see that each maximal clique of $\tilde{G}$ belongs to one of the following three classes:

*Partition cliques* Each partition clique is maximal, and such cliques correspond to multistars of $H$ and, more important, to cliques of $G$.

*3-Classes cliques* Another class of maximal cliques of $\tilde{G}$ are those" induced" by three (different) classes $P_j, P_\ell, P_k \in P_\ell$ such that there exist $\tilde{H}_a^3, \tilde{H}_b^3, \tilde{H}_c^3$ with the extremities of $\tilde{H}_a^3$ in $P_j$ and $P_\ell$, the extremities of $\tilde{H}_b^3$ in $P_\ell$ and $P_k$, the extremities of $\tilde{H}_c^3$ in $P_k$ and $P_j$: these cliques correspond to multitriangles of $H$ induced by vertices $x_j, x_\ell, x_k$, and we show in the following that they also correspond to cliques of $G$.

*Fake cliques* The last class of maximal cliques of $\tilde{G}$ are those arising from the clique $\{u_2^i, u_3^i\}$ for some strip $\tilde{H}_i^2$: note that $\{u_2^i, u_3^i\}$ is maximal unless its extremities belong to classes $P_j$ and $P_\ell \in \mathcal{P}'$ and there exist a strip $\tilde{H}_a^3$ whose extremities are in the same classes $P_j$ and $P_\ell$: in this case, $\{u_2^i, u_3^i, u_2^a\}$ is a larger maximal clique. For these cliques things are more involved: with respect to $H$, the correspondence is to a multistar centered at $y_{j,\ell}^i$ in the former case and to a multitriangle induced by vertices $x_j, x_\ell, y_{j,\ell}^i$ in the latter; with respect to $G$, in both cases, there might not be any direct correspondence to a clique.

We now analyze more in detail 3-classes cliques. So consider a multitriangle of $H$ induced by vertices $x_j, x_\ell, x_k$. By construction, each of the strips $H_a$, $H_b$, $H_c$ is either an even strip or an even-odd strip $(G^i, \mathcal{A}^i)$, for $i \in \{a, b, c\}$. Let $a_1, a_2$ be the endpoints of an even $A_1^a$–$A_2^a$ path, and define $b_1, b_2$ and $c_1, c_2$ analogously. Then these three paths along with the edges $a_2b_1$, $b_2c_2$ and $a_1c_1$ induce an odd hole, unless the three paths have length zero, i.e, $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. That is the case when $G$ is a perfect graph. If $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$ then $A_1^a \cap A_2^a \neq \emptyset$, $A_1^b \cap A_2^b \neq \emptyset$ and $A_1^c \cap A_2^c \neq \emptyset$ and by construction $(A_1^a \cap A_2^a) \cup (A_1^b \cap A_2^b) \cup (A_1^c \cap A_2^c)$ is a clique. Moreover, $H_a$, $H_b$ and $H_c$ are even-short strips. (Note that the same argument applies to the case where there are more than 3 strips that have both extremities in the classes $P_j, P_\ell, P_k$.)

We now to fake cliques. So consider a multitriangle of $H$ induced by vertices $x_j, x_\ell, y_{j,\ell}^i$ (see above). By construction, the strip $H_i$ is an odd or even-odd strip $(G^i, \mathcal{A}^i)$, while the strip $H_a$ is an even or even-odd strip $(G^a, \mathcal{A}^a)$. Therefore, there exists an even $A_1^a$–$A_2^a$ path in $(G^a, \mathcal{A}^a)$ and an odd $A_1^i$–$A_2^i$ path in $(G^i, \mathcal{A}^i)$. Then, since $G$ is odd hole free, it follows that every even $A_1^a$–$A_2^a$ path should have length zero and every odd path in $G^i$ should be of length one. Thus $(G^i, \mathcal{A}^i)$ is an odd-short strip, $(G^a, \mathcal{A}^a)$ is an even-short strip, and $A_1^a \cap A_2^a$ is non-empty and complete to $A_1^i \cup A_2^i$.

Following the above discussion, the only cliques of $\tilde{G}$ that do not correspond to any clique of $G$ are fake cliques and they "arise" from the clique $\{u_2^i, u_3^i\}$ for some strip $\tilde{H}_i^2$. We therefore start by considering the case where there are no such cliques.

### 3.3.3 When no strip $H_i$ is replaced by the strip $\tilde{H}_i^2 = (T_2^i, \tilde{\mathcal{A}}_2^i)$.

In this case, each maximal clique of $\tilde{G}$ corresponds to a maximal clique of $G$. In particular, following the discussion in the previous section, each maximal clique $\tilde{K}$ of $\tilde{G}$ is either a partition clique or a 3-classes clique. In the former case, $\tilde{K}$ corresponds to a multistar centered at some vertex $x_j$ of $H$ and we will translate into the partition-clique $\phi(\tilde{K}) = \bigcup_{A \in P_j} A$. In the latter case, $\tilde{K}$ corresponds to a multitriangle $x_j x_\ell x_k$ of $H$ and will translate into the clique $\phi(\tilde{K}) = \bigcup_{d \in I_{j,\ell,k}} (A_1^d \cap A_2^d)$, where $I_{j,\ell,k}$ is the set of indices $d$ of even-short 2-strips $H_d = (G^d, \mathcal{A}^d)$ in the decomposition having their two extremities in two different sets in $\{P_j, P_\ell, P_k\}$.

In both cases, for each maximal clique $\tilde{K}$ in the support of $\tilde{y}$, we will set $y(\phi(\tilde{K})) = \tilde{y}(\tilde{K})$. Therefore $y$ will be a "partial" weighted clique cover $y$ of $G$ with value $\tau_{\tilde{w}}(\tilde{G})$, and we need to complete $y$ on each strip $H_i$ as to cover the "residual" weight of vertices in $H_i$. As we show in the following, this will be done building upon a suitable weighted clique cover of value $\delta_1^i$ with respect to a weight function $w^i$ defined on the vertices of $G^i$ (or, in some cases, on the vertices of the auxiliary graph $G^i_=$, see Section 3). We will deal with strips in this order: 1-strips, 2-strips replaced by $\tilde{H}^0$; 2-strips replaced by $\tilde{H}^1$; 2-strips replaced by $\tilde{H}_i^3$ that are not even-short; 2-strips replaced by $\tilde{H}_i^3$ that are even-short.

**1-strips.** Let $H_i = (G^i, \mathcal{A}^i)$ be a 1-strip. $H_i$ has been replaced by the strip $\tilde{H}_i^0 = (T_0^i, \tilde{\mathcal{A}}_0^i)$, where the graph $T_0^i$ consists of a single vertex $c^i$, and $\tilde{\mathcal{A}}_0^i = \{\{c^i\}\}$. It follows from the discussion in Section 3.3.1 that there is only one clique in the support of $\tilde{y}$ covering $\tilde{\mathcal{A}}_0^i = \{\{c^i\}\}$, and that clique corresponds in $G$ to the partition clique from the class of $\mathcal{P}$ which $A_1^i$ belongs to. Therefore, each vertex in $A_1^i$ is covered by a single clique in the support of $y$, with weight at least $\tilde{w}(c^i) = \alpha_w(G^i) - \delta_1^i$, where $\delta_1^i = \alpha_w(G^i \setminus A_1^i)$. Then we can "extend" $y$ into a MWCC of $G$, with respect to $w$, because of the following lemma:

**Lemma 8** *For some $i$ between 1 and $k$, suppose that $H_i = (G^i, \mathcal{A}^i)$ is a 1-strip. Let $b$ be greater than or equal to $\alpha_w(G^i) - \delta_1^i$ and let us define function $w^i$ mapping $V(G^i)$ to $\mathbb{R}_+$ as follows:*

$$w^i(v) = \begin{cases} w(v), & \text{if } v \in V(G^i) \setminus A_1^i, \\ \max\{0, w(v) - b\}, & \text{otherwise.} \end{cases}$$

*Then $\alpha_{w^i}(G^i) = \delta_1^i$.*

*Proof.* Let $S$ be a stable set of $G^i \setminus A_1^i$. Since $w^i(v) = w(v)$ for $v \in V(G^i) \setminus A_1^i$, it follows that $w^i(S) = w(S) \leq \alpha_w(G^i \setminus A_1^i) = \delta_1^i$, and the equality holds for any MWSS $S$ of $G^i \setminus A_1^i$. Now consider a stable set $S$ of $G^i$ containing one vertex $v \in A_1^i$. If $w^i(v) = 0$, then $w^i(S) = w(S \setminus v) + w^i(v) \leq w(S \setminus v) \leq \alpha_w(G^i \setminus A_1^i) = \delta_1^i$; if $w^i(v) > 0$, then $w^i(S) = w(S) - b \leq w(S) - \alpha_w(G^i) + \delta_1^i \leq \delta_1^i$. □

**2-strips replaced by $\tilde{H}^0$.** We now move to 2-strips that have been replaced by $\tilde{H}^0$, i.e., strips with both extremities in the same class of $\mathcal{P}$. Let $H_i = (G^i, \mathcal{A}^i)$ be such a strip. $H_i$ has been replaced by the strip $\tilde{H}_i^0 = (T_0^i, \tilde{\mathcal{A}}_0^i)$, where the graph $T_0^i$ consists of a single vertex $c^i$, and $\tilde{\mathcal{A}}_0^i = \{\{c^i\}\}$. It follows from the discussion in Section 3.3.1 that there is only one clique in the support of $\tilde{y}$ covering $\tilde{\mathcal{A}}_0^i = \{\{c^i\}\}$, and that clique corresponds in $G$ to the partition clique from the class of $\mathcal{P}$ which $A_1^i$ and $A_2^i$ belong to. Therefore, each vertex in $A_1^i \cup A_2^i$ is covered by a single clique in the support of $y$, with weight at least $\tilde{w}(c^i) = \max\{\alpha_w(G^i \setminus A_1^i), \alpha_w(G^i \setminus A_2^i), \alpha_w(G^i \setminus (A_1^i \triangle A_2^i))\} - \delta_1^i$, where $\delta_1^i = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$. Then we can "extend" $y$ into a MWCC of $G$, with respect to $w$, because of the following lemma:

**Lemma 9** *For some $i$ between 1 and $k$, suppose that $H_i = (G^i, \mathcal{A}^i)$ is a 2-strip. Let $G^i_=$ be the graph obtained from $G^i$ by adding the edges between $A_1^i$ and $A_2^i$. Let $b$ be greater than or equal to $\max\{\alpha_w(G^i \setminus A_1^i), \alpha_w(G^i \setminus A_2^i), \alpha_w(G^i \setminus (A_1^i \triangle A_2^i))\} - \delta_1^i$ and let us define function $w^i$ mapping $V(G^i)$ to $\mathbb{R}_+$ as follows:*

$$w^i(v) = \begin{cases} w(v), & \text{if } v \in V(G^i) \setminus (A_1^i \cup A_2^i), \\ \max\{0, w(v) - b\}, & \text{otherwise.} \end{cases}$$

*Then $\alpha_{w^i}(G^i_=) = \delta_1^i$. Moreover, if $G^i_=$ is perfect, any MWCC of $G^i_=$ with respect to $w^i$ does not assign strictly positive weight to the clique $A_1^i \cup A_2^i$ (and so $\alpha_{w^i}(G^i_=) = \alpha_{w^i}(G^i_= \setminus (A_1^i \cup A_2^i))$).*

*Proof.* First note that $\alpha_w(G^i_=) = \max\{\alpha_w(G^i \setminus A_1^i), \alpha_w(G^i \setminus A_2^i), \alpha_w(G^i \setminus (A_1^i \triangle A_2^i))\}$. Now, let $S$ be a maximum stable set of $G^i \setminus (A_1^i \cup A_2^i)$ with respect to $w$. Since $w^i(v) = w(v)$ for $v \in V(G^i) \setminus (A_1^i \cup A_2^i)$, then $w^i(S) = w(S) = \alpha_w(G^i \setminus (A_1^i \cup A_2^i)) = \delta_1^i$. On the other hand and by the same reason, any stable set $S$ of $G^i_=$ such that $w^i(S) > \delta_1^i$ should contain a vertex $v \in A_1^i \cup A_2^i$, such that $w^i(v) > 0$, i.e., $w^i(v) = w(v) - b$. Since it is a clique of $G^i_=$, there is at most one such vertex. So, $w^i(S) = w(S) - b \leq w(S) - \alpha_w(G^i_=) + \delta_1^i \leq \delta_1^i$. Then $\alpha_{w^i}(G^i_=) = \delta_1^i$. If $G^i_=$ is perfect, any MWCC of $G^i_=$ with respect to $w^i$ should have weight $\delta_1^i$. In particular, every clique with strictly positive weight must intersect any MWSS of $G^i \setminus (A_1^i \cup A_2^i)$. So, in any MWCC of $G^i_=$, the clique $A_1^i \cup A_2^i$ has weight zero. □

**Remark 2** *According to Lemma 9, when $H_i$ is a 2-strips with both extremities in the same class of $\mathcal{P}$, the extension of $y$ into a MWCC of $G$ requires the evaluation of a MWCC of $G^i_=$. Even though $G^i_=$ is a subgraph of $G^i$, strictly speaking it is possible that $G^i_=$ does not belong to the class of graphs $\mathcal{C}$ as $G^i$ (see the statement of Theorem 2). However, there is an interesting case where we do not need dealing with $G^i_=$. Namely, suppose that there are no two vertices $v_1 \in A_1^i$ and $v_2 \in A_2^i$ with a common neighbor in $V(G^i) \setminus (A_1^i \cup A_2^i)$: in this case, the only maximal clique that might be present in $G^i_=$ but not in $G^i$ is $A_1^i \cup A_2^i$. In this case, following the last statement of the lemma, any MWCC of $G^i_=$ (with respect to $w^i$) does not assign strictly positive weight to the clique $A_1^i \cup A_2^i$, so any MWCC of $G^i$ (with respect to $w^i$) is also a MWCC of $G^i_=$ (with respect to $w^i$).*

**2-strips replaced by $\tilde{H}^1$.** Let $H_i = (G^i, \mathcal{A}^i)$ be such a strip. $H_i$ has been replaced by the strip $\tilde{H}_i^1 = (T_1^i, \tilde{\mathcal{A}}_1^i)$, where $V(T_1^i) = \{u_1^i, u_2^i\}$, $E(T_1^i) = \emptyset$, $\tilde{\mathcal{A}}_1^i = \{\tilde{A}_1^i, \tilde{A}_2^i\}$ and $\tilde{A}_1^i = \{u_1^i\}$, $\tilde{A}_2^i = \{u_2^i\}$. It follows from the discussion in Section 3.3.1 that there is only one clique in the support of $\tilde{y}$ covering $\tilde{A}_1^i = \{u_1^i\}$, and that clique corresponds in $G$ to the partition clique from the class of $\mathcal{P}$ which $A_1^i$ belongs to, and the same holds with respect to $u_2^i$ and $A_2^i$. Therefore, each vertex in $A_1^i \setminus A_2^i$ is covered by a single clique in the support of $y$, with weight at least $\tilde{w}(u_1^i) = \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, and each vertex in $A_2^i \setminus A_1^i$ is covered by a single clique in the support of $y$, with weight at least $\tilde{w}(u_2^i) = \alpha_w(G^i \setminus A_1^i) - \delta_1^i$, where $\delta_1^i = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$. Recall that $\alpha_w(G^i \setminus A_1^i) + \alpha_w(G^i \setminus A_2^i) = \alpha_w(G^i) + \delta_1^i$, as we replaced $H_i$ with $\tilde{H}_i^1$: it follows that each vertex in $A_1^i \cap A_2^i$ is covered with weight at least $\alpha_w(G^i) - \delta_1^i$. Then we can "extend" $y$ into a MWCC of $G$, with respect to $w$, because of the following lemma:

**Lemma 10** *For some $i$ between 1 and $k$, suppose that $H_i = (G^i, \mathcal{A}^i)$ is a 2-strip. Let $b_1, b_2$ be such that $b_1 \geq \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, $b_2 \geq \alpha_w(G^i \setminus A_1^i) - \delta_1^i$, and $b_1 + b_2 \geq \alpha_w(G^i) - \delta_1^i$. Let us define function $w^i$ mapping $V(G^i)$ to $\mathbb{R}_+$ as follows:*

$$w^i(v) = \begin{cases} w(v), & \text{if } v \in V(G^i) \setminus (A_1^i \cup A_2^i), \\ \max\{0, w(v) - b_1\}, & \text{if } v \in A_1^i \setminus A_2^i, \\ \max\{0, w(v) - b_2\}, & \text{if } v \in A_2^i \setminus A_1^i, \\ \max\{0, w(v) - b_1 - b_2\}, & \text{otherwise.} \end{cases}$$

*Then $\alpha_{w^i}(G^i) = \delta_1^i$.*

*Proof.* On one hand, let $S$ be a MWSS of $G^i \setminus (A_1^i \cup A_2^i)$ with respect to $w$. Since $w^i(v) = w(v)$ for $v \in V(G^i) \setminus (A_1^i \cup A_2^i)$, then $w^i(S) = w(S) = \alpha_w(G^i \setminus (A_1^i \cup A_2^i)) = \delta_1^i$. On the other hand and by the same reason, any stable set $S$ such that $w^i(S) > \delta_1^i$ should contain a vertex $v \in A_1^i \cup A_2^i$ such that $w^i(v) > 0$. In fact, without loss of generality, we can assume that every vertex in $S$ has strictly positive weight. Now, we have four cases to consider: $S$ contains a vertex $v$ of $A_1^i$ and no vertex of $A_2^i$; $S$ contains a vertex $v$ of $A_2^i$ and no vertex of $A_1^i$; $S$ contains a vertex $v$ of $A_1^i \cap A_2^i$; or $S$ contains a vertex $v$ of $A_1^i \setminus A_2^i$ and a vertex $v'$ of $A_2^i \setminus A_1^i$. In the first case, $w^i(v) = w(v) - b_1$ and so $w^i(S) = w(S) - b_1 \leq w(S) - \alpha_w(G^i \setminus A_2^i) + \delta_1^i \leq \delta_1^i$. The second case is symmetric. In the last two cases, $w^i(S) = w(S) - b_1 - b_2 \leq w(S) - \alpha_w(G^i) + \delta_1^i \leq \delta_1^i$. □

**2-strips replaced by $\tilde{H}_i^3$ that are not even-short.** Let $H_i = (G^i, \mathcal{A}^i)$ be such a strip. $H_i$ has been replaced by the strip $\tilde{H}_i^3 = (T_3^i, \tilde{\mathcal{A}}_3^i)$, with $V(T_3^i) = \{u_1^i, u_2^i, u_3^i\}$, $E(T_3^i) = \{u_1^i u_2^i, u_2^i u_3^i\}$, $\tilde{\mathcal{A}}_3^i = \{\tilde{A}_1^i, \tilde{A}_2^i\}$ and $\tilde{A}_1^i = \{u_1^i, u_2^i\}$, $\tilde{A}_2^i = \{u_2^i, u_3^i\}$. It follows from the discussion in Section 3.3.1 that there is only one clique in the support of $\tilde{y}$ covering $u_1^i$ (resp. $u_3^i$), and that clique corresponds in $G$ to the partition clique from the class of $\mathcal{P}$ which $A_1^i$ (resp. $A_2^i$) belongs to. Therefore, as $A_1^i$ and $A_2^i$ do not intersect, each vertex in $A_1^i$ is covered by a single clique in the support of $y$, with weight $b_1$ at least $\tilde{w}(u_1^i) = \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, where $\delta_1^i = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$; analogously, each vertex in $A_2^i$ is covered by a single clique in the support of $y$, with weight $b_2$ at least $\tilde{w}(u_3^i) = \alpha_w(G^i \setminus A_1^i) - \delta_1^i$. Note also that $b_1 + b_2 \geq \tilde{w}(u_2^i) = \alpha_w(G^i) - \delta_1^i$, as the only maximal cliques of $\tilde{G}$ covering $u_2^i$ contain either $u_1^i$ or $u_3^i$. Then we can "extend" $y$ into a MWCC of $G$, with respect to $w$, because of Lemma 10 again.

**2-strips replaced by $\tilde{H}_i^3$ that are even-short.** In this case, in particular, $A_1^i \cap A_2^i \neq \emptyset$. Note that such strips might be involved in some multitriangles in the root graph of $\tilde{G}$.

Let $H_i = (G^i, \mathcal{A}^i)$ be such a strip. $H_i$ has been replaced by the strip $\tilde{H}_i^3 = (T_3^i, \tilde{\mathcal{A}}_3^i)$, with $V(T_3^i) = \{u_1^i, u_2^i, u_3^i\}$, $E(T_3^i) = \{u_1^i u_2^i, u_2^i u_3^i\}$, $\tilde{\mathcal{A}}_3^i = \{\tilde{A}_1^i, \tilde{A}_2^i\}$ and $\tilde{A}_1^i = \{u_1^i, u_2^i\}$, $\tilde{A}_2^i = \{u_2^i, u_3^i\}$. It follows from the discussion in Section 3.3.1 that there is only one clique in the support of $\tilde{y}$ covering $u_1^i$ (resp. $u_3^i$), and that clique corresponds in $G$ to the partition clique from the class of $\mathcal{P}$ which $A_1^i$ (resp. $A_2^i$) belongs to. Therefore, each vertex in $A_1^i \setminus A_2^i$ is covered by a single clique in the support of $y$, with weight $b_1$ at least $\tilde{w}(u_1^i) = \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, where $\delta_1^i = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$; analogously, each vertex in $A_2^i \setminus A_1^i$ is covered by a single clique in the support of $y$, with weight $b_2$ at least $\tilde{w}(u_3^i) = \alpha_w(G^i \setminus A_1^i) - \delta_1^i$. As for the vertices in $A_1^i \cap A_2^i$, they might be also covered, with weight $a \geq 0$, by some clique $\phi(\tilde{K})$ of $G$ arising from some 3-classes clique $\tilde{K}$ containing the vertex $u_2^i$ as, for example, those of the form $\phi(\tilde{K}) = \bigcup_{d \in I_{j,\ell,k}} (A_1^d \cap A_2^d)$, with $i \in I_{j,\ell,k}$ (recall that $I_{j,\ell,k}$ is the set of indices $d$ of even-short 2-strips $H_d = (G^d, \mathcal{A}^d)$ in the decomposition having their two extremities in two different sets in $\{P_j, P_\ell, P_k\}$). So, $b_1 + b_2 + a \geq \tilde{w}(u_2^i) = \alpha_w(G^i) - \delta_1^i$. Then we can "extend" $y$ into a MWCC of $G$, with respect to $w$, because of the following lemma:

**Lemma 11** *For some $i$ between 1 and $k$, suppose that $H_i = (G^i, \mathcal{A}^i)$ is an even-short 2-strip such that $G^i$ is perfect. Let $b_1, b_2, a$ be such that $b_1 \geq \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, $b_2 \geq \alpha_w(G^i \setminus A_1^i) - \delta_1^i$, and $a + b_1 + b_2 \geq \alpha_w(G^i) - \delta_1^i$. Let us define function $w^i$ mapping $V(G^i)$ to $\mathbb{R}_+$ as follows:*

$$w^i(v) = \begin{cases} w(v), & \text{if } v \in V(G^i) \setminus (A_1^i \cup A_2^i), \\ \max\{0, w(v) - b_1\}, & \text{if } v \in A_1^i \setminus A_2^i, \\ \max\{0, w(v) - b_2\}, & \text{if } v \in A_2^i \setminus A_1^i, \\ \max\{0, w(v) - b_1 - b_2 - a\}, & \text{otherwise.} \end{cases}$$

*Then $\alpha_{w^i}(G^i) = \delta_1^i$.*

*Proof.* On one hand, let $S$ be a MWSS of $G^i \setminus (A_1^i \cup A_2^i)$ with respect to $w$. Since $w^i(v) = w(v)$ for $v \in V(G^i) \setminus (A_1^i \cup A_2^i)$, then $w^i(S) = w(S) = \alpha_w(G^i \setminus (A_1^i \cup A_2^i)) = \delta_1^i$. On the other hand and by the same reason, any stable set $S$ such that $w^i(S) > \delta_1^i$ should contain a vertex $v \in A_1^i \cup A_2^i$ such that $w^i(v) > 0$. In fact, Without loss of generality, we can assume that every vertex in $S$ has strictly positive weight. Now, we have four cases to consider: $S$ contains a vertex $v$ of $A_1^i$ and no vertex of $A_2^i$; $S$ contains a vertex $v$ of $A_2^i$ and no vertex of $A_1^i$; $S$ contains a vertex $v$ of $A_1^i \cap A_2^i$; or $S$ contains a vertex $v$ of $A_1^i \setminus A_2^i$ and a vertex $v'$ of $A_2^i \setminus A_1^i$. In the first case, $w^i(v) = w(v) - b_1$ and so $w^i(S) = w(S) - b_1 \leq w(S) - \alpha_w(G^i \setminus A_2^i) + \delta_1^i \leq \delta_1^i$. The second case is symmetric. In the third case, $w^i(v) = w(v) - b_1 - b_2 - a$, and so $w^i(S) = w(S) - b_1 - b_2 - a \leq w(S) - \alpha_w(G^i) + \delta_1^i \leq \delta_1^i$. In the last case, $w^i(v) = w(v) - b_1$ and $w^i(v') = w(v') - b_2$, and so $w^i(S) = w(S) - b_1 - b_2 \leq w(S) - \alpha_w(G^i \setminus A_2^i) - \alpha_w(G^i \setminus A_1^i) + 2\delta_1^i$. Note that, since $H_i$ is an even-short strip, the strip $H_i' = (G^i \setminus (A_1^i \cap A_2^i), \{A_1^i \setminus A_2^i, A_2^i \setminus A_1^i\})$ is either non-connected or odd, so by Lemma 6, $\alpha_w(G^i \setminus A_1^i) + \alpha_w(G^i \setminus A_2^i) \geq \alpha_w(G^i \setminus (A_1^i \cap A_2^i)) + \delta_1^i$. Thus $w^i(S) \leq w(S) - \alpha_w(G^i \setminus (A_1^i \cap A_2^i)) + \delta_1^i \leq \delta_1^i$. □

### 3.3.4 When some strip is replaced by the strip $\tilde{H}_i^2 = (T_2^i, \tilde{\mathcal{A}}_2^i)$.

We finally deal with the case where some strip $H_i$ has been replaced by the strip $\tilde{H}_i^2 = (T_2^i, \tilde{\mathcal{A}}_2^i)$.

Let $H_i = (G^i, \mathcal{A}^i)$ be such a strip. $H_i$ has been replaced by the strip $\tilde{H}_i^2 = (T_2^i, \tilde{\mathcal{A}}_2^i)$, with $V(T_2^i) = \{u_1^i, u_2^i, u_3^i\}$, $E(T_2^i) = \{u_1^i u_2^i, u_2^i u_3^i\}$, $\tilde{\mathcal{A}}_2^i = \{\tilde{A}_1^i, \tilde{A}_2^i\}$ and $\tilde{A}_1^i = \{u_1^i, u_2^i\}$, $\tilde{A}_2^i = \{u_3^i\}$. We also set: $\tilde{w}(u_1^i) = \alpha_w(G^i) - \alpha_w(G^i \setminus A_1^i)$, $\tilde{w}(u_2^i) = \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, $\tilde{w}(u_3^i) = \alpha_w(G^i \setminus A_1^i) - \delta_1^i$.

Recall that, in this case, there might be some maximal cliques of $\tilde{G}$ that do not correspond to any maximal clique of $G$, we called these cliques fake. We will show how to define a weight function on the vertices of the strip $H_i$ so as to get a cover with the same value which includes only cliques.

Recall that these fake cliques correspond in the root graph $H$ to some multistar centered at $y_{j,\ell}^i$ and to some multitriangle $y_{j,\ell}^i x_j x_\ell$. We first deal with the case where in $H$ there is a multitriangle $y_{j,\ell}^i x_j x_\ell$, as the other case will follow easily. The edge $x_j x_\ell$ in the root graph of $\tilde{G}$ implies the existence of at least one 2-strip $(G^a, \mathcal{A}^a)$ whose extremities belong to $P_j$ and $P_\ell$ and that has been replaced by the gadget strip $\tilde{H}_a^3$. Following our discussion in Section 3.3.2, $(G^i, \mathcal{A}^i)$ is odd-short and $(G^a, \mathcal{A}^a)$ is even-short, thus there is at least one vertex $x \in A_1^a \cap A_2^a$

that is complete to $A_1^i \cup A_2^i$. Without loss of generality, we assume that $x$ is unique (in fact, if there are more vertices from $(G^a, \mathcal{A}^a)$ or some other 2-strips satisfying the same properties, then they form a clique of $G$). We prove the following lemma, which essentially shows that, if our cover of $\tilde{G}$ has assigned a weight $a > 0$ to the clique corresponding to the multitriangle $y_{j,\ell}^i x_j x_\ell$, then we can discard this clique and ask for a MWCC of value $\delta_1^i + a$ in the graph induced by $G^i$ and $x$, that we denote by $G_\bullet^i$ and is trivially perfect, in such a way that $x$ is covered by a quantity greater or equal to $a$.

**Lemma 12** *For some $i$ between 1 and $k$, suppose that $H_i = (G^i, \mathcal{A}^i)$ is a 2-strip. Let $G_\bullet^i$ be the graph obtained from $G^i$ by adding a new vertex $x$ complete to both $A_1^i$ and $A_2^i$. Let $b_1, b_2, a$ be such that $b_1 \geq \alpha_w(G^i) - \alpha_w(G^i \setminus A_1^i)$, $a + b_1 \geq \alpha_w(G^i \setminus A_2^i) - \delta_1^i$, $a + b_2 \geq \alpha_w(G^i \setminus A_1^i) - \delta_1^i$. Let us define function $w^i$ mapping $V(G_\bullet^i)$ to $\mathbb{R}_+$ as follows:*

$$w^i(v) = \begin{cases} w(v), & \text{if } v \in V(G^i) \setminus (A_1^i \cup A_2^i), \\ \max\{0, w(v) - b_1\}, & \text{if } v \in A_1^i \setminus A_2^i, \\ \max\{0, w(v) - b_2\}, & \text{if } v \in A_2^i \setminus A_1^i, \\ \max\{0, w(v) - b_1 - b_2\}, & \text{if } v \in A_1^i \cap A_2^i, \\ a, & \text{otherwise (i.e., if } v = x). \end{cases}$$

*Then $\alpha_{w^i}(G_\bullet^i) = \delta_1^i + a$. In particular, $\alpha_{w^i}(G^i) \leq \delta_1^i + a$.*

*Proof.* On one hand, let $S$ be a MWSS of $G^i \setminus (A_1^i \cup A_2^i)$ with respect to $w$. Then $S \cup \{x\}$ is a stable set of $G_\bullet^i$. Since $w^i(v) = w(v)$ for $v \in V(G^i) \setminus (A_1^i \cup A_2^i)$, then $w^i(S \cup \{x\}) = w(S) + w^i(x) = \alpha_w(G^i \setminus (A_1^i \cup A_2^i)) + a = \delta_1^i + a$. In fact, since $x$ is complete to $A_1^i \cup A_2^i$ in $G_\bullet^i$, any stable set of $G_\bullet^i$ containing $x$ should be composed by $x$ and a stable set of $G^i \setminus (A_1^i \cup A_2^i)$, and will have weight $w^i$ at most $\delta_1^i + a$. So, any stable set $S$ such that $w^i(S) > \delta_1^i + a$ should contain a vertex $v \in A_1^i \cup A_2^i$ such that $w^i(v) > 0$. In fact, without loss of generality, we can assume that every vertex in $S$ has strictly positive weight. Now, we have four cases to consider: $S$ contains a vertex $v$ of $A_1^i$ and no vertex of $A_2^i$; $S$ contains a vertex $v$ of $A_2^i$ and no vertex of $A_1^i$; $S$ contains a vertex $v$ of $A_1^i \setminus A_2^i$ and a vertex $v'$ of $A_2^i \setminus A_1^i$; or $S$ contains a vertex $v$ of $A_1^i \cap A_2^i$. In the first case, $w^i(v) = w(v) - b_1$ and so $w^i(S) = w(S) - b_1 \leq w(S) - \alpha_w(G^i \setminus A_2^i) + \delta_1^i + a \leq \delta_1^i + a$. The second case is symmetric. In the third case, $w^i(v) = w(v) - b_1$ and $w^i(v') = w(v') - b_2$, and in the last case $w^i(v) = w(v) - b_1 - b_2$. So, in both cases, $w^i(S) = w(S) - b_1 - b_2$. By adding the first two required inequalities, it follows that $a + b_1 + b_2 \geq \alpha_w(G^i) - \delta_1^i$, so $w^i(S) \leq w(S) - \alpha_w(G^i) + \delta_1^i + a \leq \delta_1^i + a$. □

Note that, when $\alpha_w(G^i \setminus A_1^i) + \alpha_w(G^i \setminus A_2^i) = \alpha_w(G^i) + \delta_1^i$, the conditions $b_1 \geq \alpha_w(G^i \setminus A_2^i) - \delta_1^i$ and $b_2 \geq \alpha_w(G^i \setminus A_1^i) - \delta_1^i$, imply that $b_1 + b_2 \geq \alpha_w(G^i) - \delta_1^i$.

We observe that the last sentence of Lemma 12, i.e., the fact that $\alpha_{w^i}(G^i) \leq \delta_1^i + a$, suggests also how to "translate" the weight $a$ possibly assigned to a clique of $\tilde{G}$ corresponding to the multistar centered at some $y_{j,\ell}^i$. Note also that in this case we do *not* need to evaluate a MWCC of $G_\bullet^i$. Therefore, we need a MWCC of $G_\bullet^i$ only when there exists $x \in A_1^a \cap A_2^a$ that is complete to $A_1^i \cup A_2^i$: as in this case $G_\bullet^i$ is an induced subgraph of $G$.

We are done with the proof of Theorem 2. Following Remark 2 and the discussion above, we are indeed able to provide a slightly technical improvement for it, which will be useful for the following:

**Theorem 13** *Let $\mathcal{G}$ be the class of perfect graphs which are the composition of strips $H_i = (G^i, \mathcal{A}^i)$ $i = 1, \ldots, k$ with respect to a partition $\mathcal{P}$, such that each of the strips belongs to a class $\mathcal{C}$. Suppose also that there exists a function $p$ such that for a strip $H_i = (G^i, \mathcal{A}^i)$ in $\mathcal{C}$, we can compute in time $O(p(|V(G^i)|))$:*

*(i) a MWCC of $G^i$ and of $G^i_\bullet$, if $H_i$ is an odd-short strip and $G^i_\bullet$ is an induced subgraph of $G$;*

*(ii) a MWCC of $G^i$ and of $G^i_=$, if $G^i_=$ is an induced subgraph of $G$, $A^i_1$ and in $A^i_2$ belong to the same class of $\mathcal{P}$, and there are two vertices $v_1 \in A^i_1$ and $v_2 \in A^i_2$ with a common neighbor in $V(G^i) \setminus (A^i_1 \cup A^i_2)$. In particular, for the MWCC on $G^i_=$, it is enough to deal with weight functions $w^i$ defined on $V(G^i_=)$ such that $\alpha_{w^i}(G^i_=) = \alpha_{w^i}(G^i_= \setminus (A^i_1 \cup A^i_2))$;*

*(iii) a MWCC of $G^i$ otherwise.*

*Then the MWCC problem on $G \in \mathcal{G}$ can be solved in time $O(\sum_{i=1}^k p(|V(G^i)|) + match(|V(G)|))$, where $match(n)$ is the time required to solve the matching problem on a graph with $n$ vertices. If $p$ is a polynomial, then the MWCC can be solved on the class $\mathcal{G}$ in polynomial time.*

# 4 The MWCC on claw-free perfect graphs.

As an application of Theorem 13, we give a new algorithm for the MWCC on strip-composed claw-free perfect graphs. Recall that claw-free perfect graphs are in fact quasi-line. In the last decade the structure of quasi-line graphs was deeply investigated, with some results providing a detailed description and characterization of the strips that, through composition, can be part of a quasi-line graph. This is the case of the structure theorem by Chudnovsky and Seymour in [6] and the structure theorem by Chudnovsky and Plumettaz in [4].

An algorithmic decomposition theorem for quasi-line graph has been given in [10, 11]. Before stating this theorem, we need a few definitions. The first one is that of *net*, i.e., a graph formed by a triangle and three vertices of degree one, each of them adjacent to a distinct vertex of the triangle.

**Definition 14** *A clique $K$ of a connected graph $G$ is* distance simplicial *if, for every $j$, its $j$-th neighborhood is also a clique. In this case, we also say that $G$ is* distance simplicial *with respect to $K$ (or simply distance simplicial). Finally, a strip $H = (G, \mathcal{A})$ is distance simplicial if $G$ is distance simplicial with respect to each $A \in \mathcal{A}$ and:*

*(j) either $A^i_1 = A^i_2 = V(G^i)$;*

*(jj) or $A^i_1 \cap A^i_2 = \emptyset$ and there exists $j_2$ such that $N_{j_2+1}(A^i_1) = \emptyset$, $N_{j_2}(A^i_1) \cap A^i_2 \neq \emptyset$, $N_{j_2-1}(A^i_1) \cup N_{j_2}(A^i_1) \supseteq A^i_2$, where $N_j(A^i_1)$ is the $j$-th neighborhood of $A^i_1$ in $G^i$ (and a similar statement holds with respect to $A^i_2$).*

**Theorem 15** *[10, 11] Let $G$ be a connected quasi-line graph. In time $O(|V(G)||E(G)|)$, one can:*

*(i) either recognize that $G$ is net-free;*

*(ii) or provide a decomposition of $G$ into $k \leq |V(G)|$ quasi-line strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$, such that each graph $G^i$ is distance simplicial with respect to each clique $A \in \mathcal{A}^i$. Moreover, if $A^i_1 \cap A^i_2 = \emptyset$, then:*
- *each vertex in $A$ has a neighbor in $V(G^i) \setminus A$, for each $A \in \mathcal{A}^i$;*
- *if $A^i_1$ and $A^i_2$ are in the same set of $\mathcal{P}$, then $A^i_1$ is anticomplete to $A^i_2$.*

### 4.1 The MWCC problem on strip composed claw-free perfect graphs.

Suppose that we are given a claw-free *perfect* graph $G$ that is the strip composition of strips obeying to case $(ii)$ of Theorem 15. If we are interested in finding a MWCC of $G$, following Theorem 13, we must show that for a strip that is distance simplicial we can compute in polynomial time:

1. a MWCC of the strip;
2. a MWCC of $G^i_\bullet$, i.e., $G^i$ plus a vertex complete to both extremities, when the strip $(G^i, \mathcal{A}^i)$ is odd-short and $G^i_\bullet$ is an induced subgraph of $G$: note that in this case $(G^i, \mathcal{A}^i)$ must obey to $(jj)$ in Definition 14, as otherwise the strip is even since $A^i_1 = A^i_2 = V(G^i)$;
3. a MWCC of $G^i_=$, i.e., $G^i$ plus edges as to make $A^i_1$ complete to $A^i_2$, if $G^i_=$ is an induced subgraph of $G$, $A^i_1$ and in $A^i_2$ belong to the same class of $\mathcal{P}$, and there are two vertices $v_1 \in A^i_1$ and $v_2 \in A^i_2$ with a common neighbor in $V(G^i) \setminus (A^i_1 \cup A^i_2)$: note that also in this case $(G^i, \mathcal{A}^i)$ must obey to $(jj)$ in Definition 14, as otherwise $V(G^i) \setminus (A^i_1 \cup A^i_2) = \emptyset$.

Before going into each case, we give a last definition. A graph is *cobipartite* if its vertex set can be covered by two cliques. Note that a cobipartite graph is distance simplicial with respect to each of the two cliques covering its vertex set. Also it is not difficult to see that distance simplicial graphs are perfect, since they can be iteratively decomposed by clique cutsets into cobipartite graphs.

**1.** We start by briefly describing how to compute a MWCC in distance simplicial graphs (we just observed that they are perfect). We rely on a property of perfect graphs, namely, there always exists a clique which intersects each MWSS: we will call such a clique *crucial* (crucial cliques are a key ingredient to the algorithm in [23]). Our algorithm relies on the fact that for graphs that are distance simplicial with respect to some identifiable clique $K$, we can inductively compute crucial cliques and decide the value of this clique in a MWCC. The first crucial clique will be $K' := K \cup \{v \notin K : v \text{ is complete to } K\}$: we will suitably update the weight of each vertex, and then find a new crucial clique (with respect to the new weights) in an inductive way.

Assume therefore that $D$ is a distance simplicial graph with respect to a clique $K_1$ and let $w$ be a strictly positive weight function on $V(D)$. In the following, for the sake of shortness, we let $K_{j+1} := N_j(K_1)$.

**Lemma 16** *Algorithm 1 is correct and can be implemented as to run in* $O(|V(D)|^2)$*-time.*

*Proof.* We claim the following property. Let $Q \subseteq V(D)$ be a non-empty subset of vertices. Let $j \in \{1, \ldots, t+1\}$ be such that $K_i \cap Q = \emptyset$ for every $1 \le i < j$, and $K_j \cap Q \neq \emptyset$. Then, in $D[Q]$, $(K_j \cap Q) \cup \{v \in Q \setminus K_j : v \text{ is complete to } K_j \cap Q \text{ in the graph } D[Q]\}$ is a crucial clique.

*Proof of the claim.* Since $D$ is distance simplicial with respect to $K_1$, $(K_j \cap Q) \cup \{v \in Q \setminus K_j : v \text{ is complete to } K_j \cap Q \text{ in the graph } D[Q]\}$ is a clique in $D[Q]$. Suppose that there is a MWSS $S$ in $D[Q]$ that does not intersect it. In particular, $j < t+1$, no vertex of $S$ belongs to $K_j$, and no vertex of $S$ is complete to $K_j \cap Q$. Since $K_{j+1}$ is a clique, at most one vertex of $S$ belongs to it, and any other vertex of $S$ is anticomplete to $K_j$. In any case, there is a vertex in $K_j \cap Q$ that is anticomplete to $S$, a contradiction to the maximality of $S$, since the weight $w$ is strictly positive. This proves the claim.

**Algorithm 1**

**Require:** A graph $D$ that is distance simplicial graph with respect to a clique $K_1$ and a strictly positive weight function $w$ on $V(D)$. Assume that $K_{j+1} \neq \emptyset$, for every $1 \leq j \leq t$, while $K_{j+2} = \emptyset$.

**Ensure:** A MWCC for $D$ with respect to the weight function $w$.

1: Let $Q \leftarrow V(D)$; $y = 0$;

2: While $Q \neq \emptyset$ do:

  **2.1** Let $j \in \{1, \ldots, t\}$ be such that $K_i \cap Q = \emptyset$ for $1 \leq i < j$ and $K_j \cap Q \neq \emptyset$.

  **2.2** Let $K \leftarrow (K_j \cap Q) \cup \{v \in Q \setminus K_j : v$ is complete to $K_j \cap Q$ in the graph $D[Q]\}$.

  **2.3** Let $\bar{v}$ be the vertex of $K$ with minimum (current) weight $w$.

  **2.4** Let $Q \leftarrow Q \setminus \{v \in K : w(v) = w(\bar{v})\}$.

  **2.5** For each $v \in K$, let $w(v) \leftarrow w(v) - w(\bar{v})$.

  **2.6** Let $y(K) \leftarrow w(\bar{v})$.

3: **Return** $y$.

By the claim, the set $K$ we build at step 2.2 is a clique that intersects every MWSS of the current graph. In steps 2.5 and 2.6, we are decreasing the weighted stability number of the current graph by $y(K)$, or in other words $\sum_{K \in \mathcal{K}(D)} y(K) = \alpha_w(D)$ (where $\mathcal{K}(D)$ is the collection of all the cliques of the graph $D$). In fact let us call $w'$ the weight function after step 2.5 and suppose by contradiction that $\alpha_{w'}(D) > \alpha_w(D) - \bar{w}$, and denote with $S'$ the maximum weight stable set with respect to the weight function $w'$ and with $D' := D[Q]$ after step 2.4. We have to analyze two cases: $(i)$ $S' \cap K \neq \emptyset$ and $(ii)$ $S' \cap K = \emptyset$. If $(i)$ holds then $\alpha_{w'}(D) = w'(S') = w(S') - \bar{w} \leq \alpha_w(D) - \bar{w}$ which is a contradiction. If $(ii)$ holds we know that in $D'$, $\bar{K} = \bar{K}_j \cup \{v \notin \bar{K}_j : v$ is complete to $\bar{K}_j$ in the graph $D[Q]\}$, where $\bar{K}_j$ is $K_j$ restricted to vertices with strictly positive weight, is a crucial clique, so in particular $S' \cap \bar{K} \neq \emptyset$. Since $S' \cap K = \emptyset$ and $S' \cap \bar{K} \neq \emptyset$, we have that $S'$ contains a vertex $x \notin \bar{K}_j$ such that $x$ is complete to $\bar{K}_j$ in the graph $D[Q]$ that in $D$ was not complete to $K_j$, or in other words, $x$ was not adjacent to some vertex of $y \in K_j$ of weight $\bar{w}$. But then we can consider the set $S' \cup \{y\}$ and we can observe that this is a stable set in $D$, and its weight is $w'(S') + \bar{w} = \alpha_{w'}(D) + \bar{w} > \alpha_w(D)$, which is a contradiction.

Moreover, as the stop condition for step 2 is $Q = \emptyset$, we have covered every vertex with its weight and this concludes correctness.

In the uniform cost model, steps 2.1 to 2.6 can be implemented as to run in $O(|V(D)|)$-time, and we can easily observe that they will be repeated at most $|V(D)|$ because each time we perform step 2.4 the cardinality of the set $Q$ strictly decreases. □

**2.** Consider now an odd-short distance simplicial strip $H_i = (G^i, \mathcal{A}^i)$ such that:

- $G^i_\bullet$ is an induced subgraph of $G$, and therefore claw-free perfect;
- $A^i_1 \cap A^i_2 = \emptyset$ and there exists $j_2$ such that $N_{j_2+1}(A^i_1) = \emptyset$, $N_{j_2}(A^i_1) \cap A^i_2 \neq \emptyset$, $N_{j_2-1}(A^i_1) \cup N_{j_2}(A^i_1) \supseteq A^i_2$ (and a similar statement holds with respect to $A^i_2$)

**Lemma 17** *$G^i_\bullet$ is cobipartite.*

*Proof.* Recall that $G^i_\bullet$ is $G^i$ plus a vertex $x$ complete to both extremities. Since $G^i$ is odd-short, then $A^i_2 \cap N(A^i_1) \neq \emptyset$. It follows that $A^i_2 \subseteq N(A^i_1) \cup N_2(A^i_1)$. If $N_2(A^i_1)$ is empty, then $(A^i_1 \cup \{x\}, N(A^i_1))$ is a bipartition of $\overline{G^i_\bullet}$. The same holds if $N_2(A^i_2)$ is empty. So, suppose that $N_2(A^i_1)$ and $N_2(A^i_2)$ are both non-empty.

We claim that: *(i)* $N(A_1^i) \setminus A_2^i = N(A_2^i) \setminus A_1^i$; *(ii)* $N_2(A_1^i) \setminus (A_1^i \cup A_2^i) = \emptyset$. *Proof of Claim (i).* Let $v \in N(A_1^i) \setminus A_2^i$ then, since $G^i$ is distance simplicial with respect to $A_1^i$, $v$ is complete to $A_2^i \cap N(A_1^i)$, that is non-empty. And so, $v \in N(A_2^i)$. Symmetrically, every vertex in $N(A_2^i) \setminus A_1^i$ belongs to $N(A_1^i)$, and that proves the claim. *Proof of Claim (ii).* Suppose there is a vertex $v \in N_2(A_1^i) \setminus A_2^i$. Then, since $G^i$ is distance simplicial with respect to $A_1^i$, $v$ is complete to $A_2^i \cap N_2(A_1^i)$, that is non-empty. And so, $v \in N(A_2^i)$. But, by claim (i) $v$ would then belong to $N(A_1^i)$, a contradiction.

Claims (i) and (ii) imply that $B = V(G^i) \setminus (A_1^i \cup A_2^i) \subseteq N(A_1^i)$ is a clique.

The vertices of $G^i_\bullet$ can be partitioned into four cliques, namely $A_1^i$, $A_2^i$, $\{x\}$, and $B$, such that $\{x\}$ is complete to $A_1^i \cup A_2^i$, and $B$ is complete to $(N(A_1^i) \cap A_2^i) \cup (N(A_2^i) \cap A_1^i)$. Moreover, by Theorem 15, each vertex in $(N_2(A_1^i) \cap A_2^i) \cup (N_2(A_2^i) \cap A_1^i)$ has a neighbor in $B$. In particular, since $N_2(A_1^i) \cap A_2^i$ is non-empty, $B$ is non-empty.

Since $G^i_\bullet$ is perfect, in order to prove that it is cobipartite, it is enough to prove that it has no stable set of size three. Since the non-neighbors of $x$ form a clique, if there is a stable set of size 3, then it has one vertex in each of $A_1^i \cap N_2(A_2^i)$, $A_2^i \cap N_2(A_1^i)$, and $B$. Let $v, v'$ be two non-adjacent vertices in $A_1^i$ and $A_2^i$, respectively. Then, they cannot have both a common neighbor and a common non-neighbor in $B$. To the contrary, let $w$ be a common neighbor and $w'$ a common non-neighbor of $v, v'$ in $B$. Since $B$ is a clique, $w, w', v, v'$ induce a claw in $G^i$, a contradiction. Suppose that $v, v'$ have a common non-neighbor in $B$. Since they have also at least one neighbor each in $B$, and they do not have a common neighbor, there exist $w, w' \in B$ such that $w$ is adjacent to $v$ and not to $v'$ and $w'$ is adjacent to $v'$ and not to $v$. But then $vww'v'x$ induce a hole of length five on $G^i_\bullet$, a contradiction. So, there is no stable set of size three in $G^i_\bullet$, and it is cobipartite. □

**3.** Consider now a distance simplicial strip $H_i = (G^i, \mathcal{A}^i)$ such that:
- $G^i_=$ is an induced subgraph of $G$, and therefore claw-free perfect;
- $A_1^i \cap A_2^i = \emptyset$ and $A_1^i$ and in $A_2^i$ belong to the same class of $\mathcal{P}$, and there are two vertices $v_1 \in A_1^i$ and $v_2 \in A_2^i$ with a common neighbor in $V(G^i) \setminus (A_1^i \cup A_2^i)$;
- $\alpha_{w^i}(G^i_=) = \alpha_{w^i}(G^i_= \setminus (A_1^i \cup A_2^i))$ holds, where $w^i$ is the weight function defined on the vertices of $G^i$ (that without loss of generality we take strictly positive, i.e., we remove vertices with $w^i(v) = 0$)

**Lemma 18** *Either $G^i_=$ is cobipartite, or every* MWCC *of $G^i$ is also a* MWCC *of $G^i_=$.*

*Proof.* We claim that $V(G^i) \setminus (A_1^i \cup A_2^i)$ can be partitioned into three complete sets, namely $B = (N(A_1^i) \setminus A_2^i) \cap (N(A_2^i) \setminus A_1^i)$, $C_1 = N(A_1^i) \setminus (A_2^i \cup N(A_2^i))$ and $C_2 = N(A_2^i) \setminus (A_1^i \cup N(A_1^i))$. Moreover, $B$ is complete to $C_1 \cup C_2$, $A_1^i$ is anticomplete to $C_2$ and $A_2^i$ is anticomplete to $C_1$.

*Proof of the claim.* Let us consider now the graph $G^i$ that, by Theorem 15, is distance simplicial with respect to $A_1^i$ and $A_2^i$ and in which, by the same theorem, $A_1^i$ is anticomplete to $A_2^i$. By hypothesis, there are two vertices $v_1 \in A_1^i$ and $v_2 \in A_2^i$ having a common neighbor in $V(G^i) \setminus (A_1^i \cup A_2^i)$. So, $B = N^{G^i}(A_1^i) \cap N^{G^i}(A_2^i) = (N^{G^i_=}(A_1^i) \setminus A_2^i) \cap (N^{G^i_=}(A_2^i) \setminus A_1^i)$ is non-empty. This implies that there is a vertex in $A_2^i \cap N_2^{G^i}(A_1^i)$ and, by Theorem 15, ${N_4}^{G^i}(A_1^i)$ is empty. Symmetrically, ${N_4}^{G^i}(A_2^i)$ is empty. Let $C_1 = N^{G^i}(A_1^i) \setminus N^{G^i}(A_2^i) = N^{G^i_=}(A_1^i) \setminus (A_2^i \cup N^{G^i_=}(A_2^i))$ and $C_2 = N^{G^i}(A_2^i) \setminus N^{G^i}(A_1^i) = N^{G^i_=}(A_2^i) \setminus (A_1^i \cup N^{G^i_=}(A_1^i))$. Since $G^i$ is distance simplicial with respect to $A_1^i$ and $A_2^i$, $B$ is a clique and it is complete to $C_1$ and $C_2$. Moreover, $N^{G^i}(A_1^i) C_1 \cup B$, and $N^{G^i}(A_2^i) = C_2 \cup B$. Since $B$ is non-empty, $A_2^i \cap N_2^{G^i}(A_1^i)$

is non-empty. Since $N_2^{G^i}(A_1^i)$ is a clique, $N_2^{G^i}(A_1^i) \subseteq (A_2^i \cup N^{G^i}(A_2^i)) \setminus N^{G^i}(A_1^i) = A_2^i \cup C_2$. Symmetrically, $N_2^{G^i}(A_2^i) \subseteq (A_1^i \cup N^{G^i}(A_1^i)) \setminus N^{G^i}(A_2^i) = A_1^i \cup C_1$. Suppose that $N_3^{G^i}(A_1^i)$ is non-empty, and let $v \in N_3^{G^i}(A_1^i)$. Then $v$ has a neighbor in $N_2^{G^i}(A_1^i) \subseteq A_2^i \cup N^{G^i}(A_2^i)$, thus $v \in A_2^i \cup N^{G^i}(A_2^i) \cup N_2^{G^i}(A_2^i) \subseteq A_2^i \cup C_2 \cup B \cup A_1^i \cup C_1 \subseteq A_1^i \cup N^{G^i}(A_1^i) \cup N_2^{G^i}(A_1^i)$, a contradiction. Therefore, $N_3^{G^i}(A_1^i)$ and $N_3^{G^i}(A_2^i)$ are empty, and the claim holds.

Suppose that there is no MWCC of $G^i$ that is also a MWCC of $G^i_=$. Then, every MWCC of $G^i_=$ contains a clique $\mathcal{C}$ that is not a clique of $G^i$, thus, it intersects both $A_1^i$ and $A_2^i$ and, since $A_1^i$ is anticomplete to $C_2$ and $A_2^i$ is anticomplete to $C_1$, $\mathcal{C} \subseteq A_1^i \cup A_2^i \cup B$. Since $\mathcal{C}$ is a crucial clique of $G^i_=$ (it has positive weight in a MWCC of $G^i_=$), $\mathcal{C}$ intersects every maximum weight stable set of $G^i_=$. In particular, since $\alpha_w(G^i_=) = \alpha_w(G^i \setminus (A_1^i \cup A_2^i))$, it intersects every maximum weight stable set of $G^i \setminus (A_1^i \cup A_2^i)$. So, there is a maximum stable set $S$ of $G^i \setminus (A_1^i \cup A_2^i)$ such that $S \subseteq B$, namely, $S = \{b\}$, with $b \in B$. Since $\{b\}$ is also a maximum stable set of $G^i_=$ and $w$ is strictly positive, $b$ is complete to $V(G^i_=) \setminus \{b\}$. Finally, a quasi-line graph containing a universal vertex is cobipartite. □

We have therefore the following theorem for strip-composed claw-free perfect graphs. We underline that the resulting algorithm never requires the computation of any MWSS on the strips, while it uses a primal-dual algorithm for the maximum weighted matching on the root graph of $\tilde{G}$ (see Section 3).

**Theorem 19** *Let $G$ be a claw-free perfect graph as in case (ii) of Theorem 15. Then we can compute a* MWCC *of $G$ with respect to $w$ in time $O(|V(G)|^3)$.*

*Proof.* From Theorem 13, we know that, given the decomposition of $G$ into strips, we can compute a MWCC of $G$ in time $O(\sum_{i=1}^k p_i(|V(G)|) + match(|V(G)|))$. For every 2-strip with extremities in different classes of $\mathcal{P}$ and for every 1-strip, from Lemma 16 $p_i(|V(G^i)|) = O(|V(G^i)|^2)$. For every 2-strip with the extremities in the same class of $\mathcal{P}$, we first need to check if $G^i_=$ is cobipartite, which takes $O(|V(G^i)| + |E(G^i_=)|)$, and then we either compute directly a MWCC of $G^i_=$ or we compute a MWCC of $G^i$, and in both cases it takes $O(|V(G^i)|^2)$. Finally, for the computation of the MWCC of $G^i_\bullet$, when needed, it takes again $O(|V(G^i)|^2)$. Then $O(\sum_{i=1}^k p_i(|V(G^i)|)) = O(|V(G)|^2)$ and the overall complexity of the algorithm for the MWCC is $O(|V(G)|^2 + |V(G)|^2 \log |V(G)|) = O(|V(G)|^2 \log |V(G)|)$ (using the primal dual algorithm for maximum weight matching by Gabow [13])). As it takes $O(|V(G)|^3)$ to obtain the decomposition in strips, this is the overall complexity bound of the algorithm. □

## 4.2 The MWCC problem on {claw,net}-free perfect graphs.

In this section, we describe an approach to the MWCC problem based on clique cutset decomposition. Following the results in the previous sections and Theorem 15, we may restrict to the case where $G$ is a {claw,net}-free perfect graph. Brandstädt and Dragan [2] indeed characterized {claw,net}-free graphs, and this structure was used in [11] to deal with the MWSS problem with the case of non-strip-composed for claw-free graphs, but we could not adapt these ideas to the MWCC. We will instead exploit the structure of clique cutset decompositions in {claw,net}-free graphs and will show that the computational complexity for MWCC is $O(|V(G)|^3)$.

A set $C$ is a *clique cutset* of a graph $G$ if $G[C]$ is complete and $G[V(G) \setminus C]$ has more connected components than $G$. If $G_1, \dots, G_k$ are the connected components of $G[V(G) \setminus C]$, then $C$ decomposes $G$ into the non-disjoint graphs $G[G_1 \cup C]$, ..., $G[G_k \cup C]$. If any of these graphs admits a clique cutset then the decomposition procedure can be continued, so we obtain a *clique cutset decomposition tree.* A leaf of such a tree, that is, a graph that has no clique cutset, is called an *atom.* Note that we can always assume that the decomposition tree is *binary,* as we may assume that the clique cutset $C$ decomposes $G$ into $G[G_1 \cup C]$ and $G[G_2 \cup \cdots \cup G_k \cup C]$. In particular, Tarjan [33] gave an $O(|V||E|)$ time algorithm to find a binary decomposition tree in which for every node that is not a leaf, one of its children is a leaf.

In 1982, Whitesides described a combinatorial polynomial-time algorithm for MWSS on hereditary classes of graphs that can be decomposed by clique cutsets into pieces in which one can solve MWSS in polynomial time in a combinatorial way [36]. Still in [36], Whitesides also describes an algorithm for MWCC on *perfect* hereditary classes of graphs that can be decomposed by clique cutsets into pieces in which one can solve MWCC in polynomial time in a combinatorial way.

However, in order to solve MWCC on claw-free perfect graphs, we will not directly rely on Whitesides' algorithm, as the resulting complexity would be $O(|V(G)|)^5$ (we skip the details). We will first show that, when dealing with {claw,net}-free perfect graphs, one may assume that the decomposition tree returned by Tarjan's algorithm satisfies some additional conditions that in practice imply that one may reduce the problem to the solution of a single MWCC problem on an atom graph. Since Chvátal and Sbihi [8] proved that for claw-free perfect graphs, atoms are either *peculiar* or *elementary* graphs, we will show how to solve the MWCC on both classes in time $O(|V(G)|)^3$. This, combined with the result from the previous section, shows that we can solve the MWCC problem in claw-free perfect graphs in time $O(|V(G)|)^3$.

### 4.2.1 Decomposition trees for {claw,net}-free perfect graphs

Let $G$ be a connected graph, $C$ be a clique cutset of $G$, and $(A, B)$ be a partition of $V(G) \setminus C$ such that $A$ is anticomplete to $B$. Without loss of generality, one may always assume that for every vertex $v$ in $C$, both $N(v) \cap A$ and $N(v) \cap B$ are non-empty.

Assume now that $G$ is {claw,net}-free and perfect. The proof of the first two claims is straightforward.

**Claim 20** *Both $N_A(v)$ and $N_B(v)$ are cliques.*

**Claim 21** *Let $w$ be a vertex of $C$. Then either $N_A(v) \subseteq N_A(w)$ or $N_B(v) \subseteq N_B(w)$ (and, symmetrically, either $N_A(w) \subseteq N_A(v)$ or $N_B(w) \subseteq N_B(v)$).*

**Claim 22** *Suppose $N_B(C)$ is not a clique. Then there are at least two vertices of $C$ complete to $A$, and $A$ is a clique.*

*Proof of the claim.* Let $b_1, b_2$ be two non-adjacent vertices in $N_B(C)$, and $v_1, v_2$ their respective neighbors in $C$. By Claim 20, $v_1$ and $v_2$ are two distinct vertices, $v_1$ is not adjacent to $b_2$, and $v_2$ is not adjacent to $b_1$. By Claim 21, $N_A(v_1) = N_A(v_2)$. Because for every vertex $v$ in $C$, both $N_A(v)$ and $N_B(v)$ are non-empty, and because of Claim 20, $N_A(v_1)$ is a non-empty clique. Let $A'$ be the connected component of $G[A]$ containing $N_A(v_1)$. If $N_A(v_1) \subsetneq A'$, being $G[A']$ connected, there exist $a_1$ in $N_A(v_1)$ and $a_2$ in $A' \setminus N_A(v_1)$ such that $a_1$ is adjacent to $a_2$. But then $\{a_2, a_1, b_1, v_1, b_2, v_2\}$ induce a net in $G$, a contradiction.

Thus $N_A(v_1) = N_A(v_2) = A'$ and, by Claim 20, $A'$ is a clique. Suppose now that $G[A]$ is not connected and let $A''$ be another connected component of $G[A]$. Since $G$ is connected, there is a vertex $a_3$ in $A''$ having a neighbor $v_3$ in $C$. By Claim 21, $N_B(v_3) = N_B(v_1)$ and also $N_B(v_3) = N_B(v_2)$, but $N_B(v_1)$ and $N_B(v_2)$ are different, a contradiction. Then $A = A'$. This proves the claim.

**Claim 23** *Suppose $N_B(C)$ is a clique, and $G[B \cup C]$ is an atom. Then $B = N_B(C)$ and, in particular, $B$ is a clique. (Notice that otherwise, $N_B(C)$ would be a clique cutset of $G[B \cup C]$).*

Recall that according to [33] there exists a binary decomposition tree in which for every node that is not a leaf, one of its children is a leaf, i.e., an atom. Based on the above claims, for {claw,net}-free graphs we can prove the following:

**Lemma 24** *Let $G$ be a connected {claw,net}-free graph. Then, there is a binary clique cutset decomposition tree of $G$ such that for every node associated with a clique cutset $C$, whose children are associated with the subgraphs $G[C \cup A]$ and $G[C \cup B]$, either $A$ is a clique or $B$ is a clique. Moreover, that tree can be builded in $O(|V(G)||E(G)|)$ time.*

*Proof.* As sketched above, without loss of generality we may assume that the clique cutset decomposition tree returned by the algorithm in [33] is such that for every node, $A$, $B$ and $C$ satisfy Claims 21–23, and either $G[A \cup C]$ or $G[B \cup C]$ is an atom. Suppose $G[B \cup C]$ is an atom. If $N_B(C)$ is a clique then, by Claim 23, $B$ is a clique. If $N_B(C)$ is not a clique then, by Claim 22, $A$ is a clique and $G[A \cup C]$ is also an atom. □

Lemma 24 suggests an algorithm for MWCC on {claw,net}-free perfect graphs. We may assume without loss of generality the graph $G$ is connected. Let us consider a clique cutset tree $T$ of $G$ satisfying the conditions of the lemma. Let $C_1, \ldots, C_k$ be the cutsets associated with the internal nodes of $T$, let $G[C_1 \cup A_1], \ldots, G[C_k \cup A_k]$ be the graphs associated with $k$ of the leaves of $T$ such that $A_1, \ldots, A_k$ are cliques, and let $G[C_k \cup B]$ be the graph associated with the remaining leaf of $T$. Because for every vertex $v$ in $C$, both $N_A(v)$ and $N_B(v)$ are non-empty, and because of Claims 22, and 23, it follows that $N(A_i) = C_i$, for $i = 1, \ldots, k$. Let $n_i$ be the number of vertices of $A_i$, $m_i$ be the number of edges from $A_i$ to $C_i$ ($i = 1, \ldots, k$), and $n_B$ be the number of vertices of $G[C_k \cup B]$. Let $G_0 = G$ and, for $i = 1, \ldots, k$, $G_i = G_{i-1} \setminus A_i$. By using the first steps of Algorithm 1, we can iteratively find crucial cliques and reduce, for $i = 1, \ldots, k$, the problem in $G_{i-1}$ to the problem in $G_i$, in $O(n_i m_i)$ time. As a last step, we need to solve the problem in $G_k$, that is an atom, and therefore either an elementary graphs or a peculiar one. We may therefore state the following:

**Theorem 25** *Suppose that there exists an algorithm to solve* MWCC *in a graph $G$ that is either elementary graphs or peculiar graphs, in $O(|V(G)|^q)$ time. Then we may solve* MWCC *in {claw,net}-free graphs in time $O(|V(G)||E(G)| + |V(G)|^q)$.*

### 4.2.2 The structure of peculiar graphs.

As we will see in the following, peculiar graphs have a very simple structure, and the MWCC problem can be solved on peculiar graphs and their induced subgraphs in a similar fashion than on distance simplicial graphs (i.e., iteratively computing crucial cliques).

A graph is called *peculiar* if it can be obtained as follows: take three, pairwise vertex-disjoint, cobipartite graphs with cobipartitions $(A_1, B_2)$, $(A_2, B_3)$, $(A_3, B_1)$, respectively, such that each of them has at least one pair of non-adjacent vertices; add all edges between every pair of such graphs; then take three cliques $K_1$, $K_2$, $K_3$ that are pairwise disjoint and disjoint from the $A_i$'s and $B_i$'s; add all the edges between $K_i$ and $A_j \cup B_j$ for $j \neq i$; there is no other edge in the graph.

Let $G$ be an induced subgraph of a peculiar graph with vertex partition $A_i, B_i, K_i$, $i = 1, 2, 3$, as in the definition. Let $w$ be a positive weight function associated with its vertices. If one of the $K_i$'s is empty, $G$ is a cobipartite graph and, in particular, distance simplicial. Otherwise, since the vertices of each $K_i$ are twins, both for MWSS and MWCC we may consider that each $K_i$ consists of just one vertex $k_i$, where $k_i$ is one of the maximum weight vertices in $K_i$ (in this and the following definitions, the maximum weight is with respect to the set, not to the whole graph). Let $A'_i$ (resp. $B'_i$) be the set of maximum weight vertices of $A_i$ (resp. $B_i$). The candidates to MWSS in $G$ are $\{k_1, k_2, k_3\}$; $\{k_i, a_i\}$ and $\{k_i, b_i\}$, with $a_i \in A'_i$, $b_i \in B'_i$, and $i = 1, 2, 3$; and $\{a, b\}$ with $a \in A_i$ and $b \in B_j$, $i \in \{1, 2, 3\}$, $j = i + 1 \mod 3$, such that either $b \in B'_j$ or $a$ is complete to $B'_j$ and, analogously, either $a \in A'_i$ or $b$ is complete to $A'_i$, otherwise the set would not be a maximum weight independent set.

For each of the following two cases, we will find a crucial clique $K$ and define its weight in the clique cover, as usual in this kind of algorithms, as $y_K = \alpha_w(G) - \alpha_w(G \setminus K)$.

($i$) Suppose first that for some $i \in \{1, 2, 3\}$, there is no MWSS contained in $K_i \cup A_i$ or in $K_i \cup B_i$. Assume there is no MWSS contained in $K_1 \cup A_1$, i.e., $w(k_1) + w(a_1) < \alpha_w(G)$, for $a_1 \in A'_1$. Then the clique $K_3 \cup B_1 \cup A_2 \cup B_2$ is a crucial clique of $G$, that can be assigned weight $\alpha_w(G) - w(k_1) - w(a_1)$. The other cases are symmetric.

($ii$) Suppose now that for every $i \in \{1, 2, 3\}$, both $\{k_i, a_i\}$ and $\{k_i, b_i\}$, with $a_i \in A'_i$ and $b_i \in B'_i$, are MWSS of $G$. Then $\sum_{i=1,2,3}(2w(k_i)+w(a_i)+w(b_i)) = 6\alpha_w(G)$. As $w(k_1)+w(k_2)+w(k_3) \leq \alpha_w(G)$, it follows that $\sum_{i=1,2,3}(w(a_i) + w(b_i)) \geq 4\alpha_w(G)$. This implies that either $w(a_1) + w(b_2) > \alpha_w(G)$, or $w(a_2) + w(b_3) > \alpha_w(G)$, or $w(a_3) + w(b_1) > \alpha_w(G)$. Note that $w(a_i) + w(b_j) > \alpha_w(G)$ implies that $A'_i$ is complete to $B'_j$. Assume $w(a_1) + w(b_2) > \alpha_w(G)$, thus $A'_1$ is complete to $B'_2$. Then the clique $K_3 \cup B_1 \cup A_2 \cup B'_2 \cup \tilde{A}_1$, where $\tilde{A}_1$ is the set of vertices of $A_1$ that are complete to $B'_2$, is a crucial clique of $G$. Indeed, as in this case $A'_1 \subseteq \tilde{A}_1$, all the sets $\{k_1, a'_1\}$ with $a'_1 \in A'_1$ are intersected by the clique, as well as all the sets $\{a, b\}$ with $a \in A_1$ and $b \in B_2$ such that either $b \in B'_2$ or $a$ is complete to $B'_2$. We can assign to that clique the weight $\min\{w(a_1) - w_a, w(b_1) - w_b\}$, where $w_a$ (resp. $w_b$) is the maximum weight of a vertex in $A_1$ (resp. $B_2$) that is not in the clique, with $w_a = 0$ (resp. $w_b = 0$) if $A_1 = \tilde{A}_1$ (resp. $B_2 = B'_2$). The other cases are symmetric.

As usual, after a crucial clique $K$ is identified and its weight $y_K$ is computed, the weight of the vertices of $K$ is modified by subtracting $y_K$, and vertices with non-positive weight are removed from the graph. Then we iterate (note that, for case ($ii$), if at some point $B_2$ becomes empty, then $\tilde{A}_1 = A_1$).

Notice that the case ($i$) may arise at most six times. As for case ($ii$), each time it applies, the sizes of the sets $A_i \setminus A'_i$, $B_i \setminus B'_i$ do not increase, and at least one of them strictly decreases. After these sets are all empty, only a constant number of steps are needed. So the number of crucial cliques is linear on the number of vertices, and the computation of each crucial clique requires at most $O(|V(G)|^2)$ time.

To recognize a peculiar graph that is not cobipartite (i.e., $K_1$, $K_2$ and $K_3$ are non-empty), it is enough to find a stable set of size three $\{k_1, k_2, k_3\}$, which for a perfect graph is obtained as a certificate of the BFS algorithm if the graph is not cobipartite. The cliques $K_1$, $K_2$ and $K_3$ should be formed by the vertices that have exactly two non-neighbors in $\{k_1, k_2, k_3\}$, and we can similarly determine the sets $A_i \cup B_i$, for $i = 1, 2, 3$. Then vertices in $A_1$ can be defined as those that have a non-neighbor in $A_2 \cup B_2$, and determining $A_2$ and $A_3$ can be done similarly. Once the candidate sets are defined, it is easy to check in $O(|V(G)|^2)$ time if they agree with the definition of peculiar graphs. So, the recognition requires at most $O(|V(G)|^2)$ time.

We therefore conclude that we can find a MWCC in a peculiar graph with $n$ vertices in $O(n^3)$ time.

#### 4.2.3 The structure of elementary graphs.

Building upon a characterization of Maffray and Reed [27], we show that elementary graphs are indeed strip-composed, and the strips are cobipartite graphs.

Elementary graphs were defined by Chvátal and Sbihi as graphs whose edges can be bicolored in such a way that every chordless path on three vertices has its two edges colored differently. We are here interested in an alternative characterization of elementary graphs due to Maffray and Reed [27], that was exploited to solve the unweighted MCC problem on this class of graphs. Building upon to this characterization, we show that elementary graphs are indeed strip-composed, and the strips are cobipartite graphs. Then, by the same arguments we used in the proof of Theorem 19, we may conclude that we can find a MWCC in an elementary graph with $n$ vertices in $O(n^3)$ time.

Maffray and Reed showed that a graph is elementary if and only if it can be obtained from the line graph $G$ of a bipartite multigraph by an"augmenting" operation that consists in replacing some particular edges of $G$ by cobipartite graphs. However, since line graphs are strip-composed, and the augmenting operation itself works on strips without destroying the strip structure, elementary graphs are also strip-composed. We give more details in the following.

An edge is *flat* if it does not lie in a triangle. Let $xy$ be a flat edge of a graph $G = (V, E)$ and $B(X, Y, E_{XY})$ be a cobipartite graph that is disjoint from $G$ and such that there is at least one edge between $X$ and $Y$. The *augmentation* of $G$ with respect to $xy$ and $B$ is the graph that arises from the union of $G - \{x, y\}$ and $B$, by adding all the edges between $X$ and $N(x) - \{y\}$ and all the edges between $Y$ and $N(y) - \{x\}$. Analogously, if we are given a matching of flat edges $M = \{x_1y_1, \dots, x_hy_h\} \subseteq E$, and $h$ pairwise disjoint cobipartite graphs $B_1(X_1, Y_1, E_1), \dots, B_h(X_h, Y_h, E_h)$ that are also disjoint from $G$, the *augmentation* of $G$ with respect to $M$ and $B_1, \dots, B_h$ is the graph obtained by augmenting first $G$ with respect to $x_1y_1$ and $B_1$, then augmenting the new graph with respect to $x_2y_2$ and $B_2$ etc.

It is easy to see that each line graph $G$ is a strip-composed graph. Namely, for each vertex $x \in V(G)$ there is 2-strip $H_x = (G^x, \mathcal{A}^x)$ where $V(G^x) = A_1^x = A_2^x = \{x\}$, and the classes of $\mathcal{P}$ are in bijection with the vertices of $H$, the root of $G$ (i.e., $G$ is the line graph of $H$): each vertex $v \in H$ corresponds to the class $\{\{e\} : e \in \delta_H(v)\}$, where $\delta_H(v)$ is the star centered at $v$. In particular, if $xy$ is flat, then $P_v := \{\{x\}, \{y\}\}$ is a class of $\mathcal{P}$ corresponding to a vertex $v$ of degree 2 in $H$; moreover, if $w$ and $z$ the neighbors of $v$ in $H$, $P_w := \{\{e\} : e \in \delta_H(w)\}$ and $P_z := \{\{e\} : e \in \delta_H(z)\}$, are also classes of $\mathcal{P}$.

Suppose now that we augment the line graph $G$ with respect to the flat edge $xy$ and $B = (X, Y, E_{XY})$. Then we get a strip decomposition of the new graph as soon as: we replace the strips $H^x$ and $H^y$ by the strip $B$ with extremities $X$ and $Y$; we eliminate the partition class $P_v$; we replace $\{x\}$ by $X$ in the class $P_w$ and $\{y\}$ by $X$ in the class $P_z$. Therefore, by iterating this argument, it follows that elementary graphs, that can be obtained from line graphs by augmenting with respect to a matching of flat edges, are strip-composed graph.

We proved that we may solve MWCC in a graph $G$ that is either elementary or peculiar in $O(|V(G)|^3)$ time. We may therefore state the following corollary of Theorem 26:

**Corollary 26** *Let $G$ be a $\{claw,net\}$-free perfect graph. Then* MWCC *can be solved in time $O(|V(G)|^3)$.*


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# A Proof of Lemma 3.

*Proof.* First we prove $\alpha_w(G) \le \alpha_{w'}(G') + \delta_1^1$. Let $A = A_1$ if $H_1$ is a 1-strip and $A = A_1 \cup A_2$ if $H_1$ is a 2-strip with the extremities in the same class of the partition $\mathcal{P}$. Since $A$ is a complete set in $G$, we can partition the stable sets $S$ of $G$ in the following way:

1) $S \cap A = \emptyset$;
2) $|S \cap A| = 1$.

In case 1), we have that $w(S) = w(S\cap(G\setminus G^1))+w(S\cap G^1) \le w(S\cap(G\setminus G^1))+\delta_1^1$, where the last inequality follows from the fact that $S$ misses $A$. Therefore, $w(S\cap(G\setminus G^1)) \ge w(S)-\delta_1^1$. Moreover, $S\cap(G\setminus G^1)$ is a stable set of $G'$ and $w'(S\cap(G\setminus G^1)) = w(S\cap(G\setminus G^1))$. It follows that $\alpha_{w'}(G') \ge w(S) - \delta_1^1$.

In case 2), we have that $w(S) = w(S\cap(G\setminus G^1))+w(S\cap G^1)$. If $H_1$ is a 1-strip, then $w(S\cap G^1) \le \alpha_w(G^1)$. If $H_1$ is a 2-strip, then, as $|S\cap A| = 1$, we have that either $S\cap A \subseteq A_1^1\cap A_2^1$, or $S\cap A \subseteq A_1^1\setminus A_2^1$, or $S\cap A \subseteq A_2^1\setminus A_1^1$. Then, either $S\cap G^1 \subseteq G^1\setminus(A_1^1\triangle A_2^1)$, or $S\cap G^1 \subseteq G^1\setminus A_2^1$, or $S\cap G^1 \subseteq G^1\setminus A_1^1$. So, $w(S\cap G^1) \le \max\{\alpha_w(G^1\setminus A_1^1), \alpha_w(G^1\setminus A_2^1), \alpha_w(G^1\setminus(A_1^1\triangle A_2^1))\}$. In this case, $S\cap(G\setminus G^1)\cup\{c^1\}$ is a stable set of $G'$, and $w'(S\cap(G\setminus G^1)\cup\{c^1\}) = w(S\cap(G\setminus G^1))+\tilde{w}(c^1)$. Then we have that $\alpha_{w'}(G') \ge w(S\cap(G\setminus G^1))+\tilde{w}(c^1) = w(S)-w(S\cap G^1)+\tilde{w}(c^1) \ge w(S)-\delta_1^1$, where the last inequality holds by the previous case analysis.

Thus we have shown that for every stable set $S$ of $G$, $\alpha_{w'}(G') \ge w(S) - \delta_1^1$. In particular, this must hold for a MWSS of $G$, so we obtain $\alpha_w(G) \le \alpha_{w'}(G') + \delta_1^1$.

Now we want to prove $\alpha_w(G) \ge \alpha_{w'}(G') + \delta_1^1$. We can partition the stable sets $S'$ of $G'$ in the following way:

1) $c^1 \notin S'$;
2) $c^1 \in S'$.

In case 1), let $S^1$ be a MWSS of $G^1\setminus A$. Then, as $S'$ misses $c^1$ and there are no edges between $G^1\setminus A$ and $G\setminus G^1$, $S^1\cup S'$ is a stable set of $G$. It follows that $\alpha_w(G) \ge w(S'\cup S^1) = w'(S')+\delta_1^1$.

In case 2), let $S^1$ be a stable set of $G^1$ of maximum weight among those having at most one vertex in $A$. Now, $S'\setminus\{c^1\}\cup S^1$ is a stable set of $G$, so it holds $\alpha_w(G) \ge w(S'\setminus\{c^1\}\cup S^1) = w'(S') - \tilde{w}(c^1) + w(S^1)$. If $H_1$ is a 1-strip, then $w(S^1) = \alpha_w(G^1)$. If $H_1$ is a 2-strip, then $w(S^1) = \max\{\alpha_w(G^1\setminus A_1^1), \alpha_w(G^1\setminus A_2^1), \alpha_w(G^1\setminus(A_1^1\triangle A_2^1))\}$. In both cases, $w(S^1) - \tilde{w}(c^1) = \delta_1^1$, so $\alpha_w(G) \ge w'(S') + \delta_1^1$.

Thus we have shown that for every stable set $S'$ of $G'$, $\alpha_w(G) \ge w'(S')+\delta_1^1$. In particular, this must hold for a MWSS of $G'$, so we obtain $\alpha_w(G) \ge \alpha_{w'}(G') + \delta_1^1$. □

# B Proof of Lemma 5.

*Proof.*

(a) First we prove $\alpha_w(G) \le \alpha_{w'}(G') + \delta_1^1$. We can partition the stable sets $S$ of $G$ in the following way:

1) $S\cap(A_1^1\cup A_2^1) = \emptyset$;
2) $|S\cap(A_1^1\cup A_2^1)| = 1$;
3) $|S\cap(A_1^1\cup A_2^1)| = 2$.

In case 1), we have that $w(S) = w(S\cap(G\setminus G^1)) + w(S\cap G^1) \le w(S\cap(G\setminus G^1)) + \delta_1^1$, where the last inequality follows from the fact that $S$ misses both $A_1^1$ and $A_2^1$. Therefore,

$w(S\cap(G\setminus G^1)) \geq w(S)-\delta_1^1$. Moreover, $S\cap(G\setminus G^1)$ is a stable set of $G'$ and $w'(S\cap(G\setminus G^1)) = w(S\cap(G\setminus G^1))$. It follows that $\alpha_{w'}(G') \geq w(S)-\delta_1^1$.

Now we analyze case 2). We suppose first that $S\cap(A_1^1\setminus A_2^1)\neq\emptyset$. Then again $w(S) = w(S\cap(G\setminus G^1)) + w(S\cap G^1) \leq w(S\cap(G\setminus G^1)) + \alpha_w(G^1\setminus A_2^1)$, where again the last inequality follows from the fact that $S$ misses $A_2^1$. Now we observe that $S\cap(G\setminus G^1)\cup\{u_1^1\}$ is a stable set of $G'$, and $w'(S\cap(G\setminus G^1)\cup\{u_1^1\}) = w(S\cap(G\setminus G^1))+\tilde{w}(u_1^1)$ so we have that $\alpha_{w'}(G') \geq w(S\cap(G\setminus G^1))+\tilde{w}(u_1^1) = w(S\cap(G\setminus G^1))+\alpha_w(G^1\setminus A_2^1)-\delta_1^1 \geq w(S)-\delta_1^1$. The case in which $S\cap(A_2^1\setminus A_1^1)\neq\emptyset$ is symmetric. We suppose now that $S\cap(A_1^1\cap A_2^1)\neq\emptyset$. Thus $w(S) = w(S\cap(G\setminus G^1)) + w(S\cap G^1) \leq w(S\cap(G\setminus G^1)) + \alpha_w(G^1)$. In this case, $S\cap(G\setminus G^1)\cup\{u_1^1,u_2^1\}$ is a stable set of $G'$, and $w'(S\cap(G\setminus G^1)\cup\{u_1^1,u_2^1\}) = w(S\cap(G\setminus G^1))+\tilde{w}(u_1^1)+\tilde{w}(u_2^1)$. Then we have that $\alpha_{w'}(G') \geq w(S\cap(G\setminus G^1))+\tilde{w}(u_1^1)+\tilde{w}(u_2^1) = w(S\cap(G\setminus G^1))+\alpha_w(G^1\setminus A_2^1)-\delta_1^1+\alpha_w(G^1\setminus A_2^1)-\delta_1^1 = w(S\cap(G\setminus G^1))+\alpha_w(G^1)-\delta_1^1 \geq w(S)-\delta_1^1$, where the last equality holds by hypothesis.

The proof of case 3) follows the same lines of the last subcase of case 2).

Thus we have shown that for every stable set $S$ of $G$, $\alpha_{w'}(G') \geq w(S)-\delta_1^1$. In particular, this must hold for a MWSS of $G$, so we obtain $\alpha_w(G) \leq \alpha_{w'}(G')+\delta_1^1$.

Now we want to prove $\alpha_w(G) \geq \alpha_{w'}(G')+\delta_1^1$. We can partition the stable sets $S'$ of $G'$ in the following way:

1) $S'\cap\{u_1^1,u_2^1\}=\emptyset$;
2) $|S'\cap\{u_1^1,u_2^1\}|=1$;
3) $|S'\cap\{u_1^1,u_2^1\}|=2$.

In case 1), let $S^1$ be a MWSS of $G^1\setminus(A_1^1\cup A_2^1)$. Then, as $S'$ misses both $u_1^1$ and $u_2^1$ and there are no edges between $G^1\setminus(A_1^1\cup A_2^1)$ and $G\setminus G^1$, $S^1\cup S'$ is a stable set of $G$. It follows that $\alpha_w(G) \geq w(S'\cup S^1) = w'(S')+\delta_1^1$.

In case 2), we suppose that $u_1^1\in S'$ and let $S^1$ be a MWSS of $G^1\setminus A_2^1$. Then, as $S'$ misses $u_2^1$, $S'\setminus\{u_1^1\}\cup S^1$ is a stable set of $G$. It follows that $\alpha_w(G) \geq w(S'\setminus\{u_1^1\}\cup S^1) = w'(S')-\tilde{w}(u_1^1)+\alpha_w(G^1\setminus A_2^1) = w'(S')-\alpha_w(G^1\setminus A_2^1)+\delta_1^1+\alpha_w(G^1\setminus A_2^1) = w'(S')+\delta_1^1$. The case in which $u_2^1\in S'$ is analogous.

In case 3), let $S^1$ be a MWSS of $G^1$. Now, $S'\setminus\{u_1^1,u_2^1\}\cup S^1$ is a stable set of $G$, so it holds $\alpha_w(G) \geq w(S'\setminus\{u_1^1,u_2^1\}\cup S^1) = w'(S')-\tilde{w}(u_1^1)-\tilde{w}(u_2^i)+\alpha_w(G^1) = w'(S')-\alpha_w(G^1\setminus A_1^1)-\alpha_w(G^1\setminus A_2^1)+2\delta_1^1+\alpha_w(G^1) = w'(S')+\delta_1^1$, where the last equality holds by hypothesis.

Thus we have shown that for every stable set $S'$ of $G'$, $\alpha_w(G) \geq w'(S')+\delta_1^1$. In particular, this must hold for a MWSS of $G'$, so we obtain $\alpha_w(G) \geq \alpha_{w'}(G')+\delta_1^1$.

(b) First we prove $\alpha_w(G) \leq \alpha_{w'}(G')+\delta_1^1$. We can partition the stable sets $S$ of $G$ in the following way:

1) $S\cap(A_1^1\cup A_2^1)=\emptyset$;
2) $|S\cap(A_1^1\cup A_2^1)|=1$;
3) $|S\cap(A_1^1\cup A_2^1)|=2$.

In case 1), we have that $w(S) = w(S\cap(G\setminus G^1))+w(S\cap G^1) \leq w(S\cap(G\setminus G^1))+\delta_1^1$, where the last inequality follows from the fact that $S$ misses both $A_1^1$ and $A_2^1$. Therefore, $w(S\cap(G\setminus G^1)) \geq w(S)-\delta_1^1$. Moreover, $S\cap(G\setminus G^1)$ is a stable set of $G'$ and $w'(S\cap(G\setminus G^1)) = w(S\cap(G\setminus G^1))$. It follows that $\alpha_{w'}(G') \geq w(S)-\delta_1^1$.

Now we analyze case 2). First, suppose that $S \cap (A_1^1 \setminus A_2^1) \neq \emptyset$. Then again $w(S) = w(S \cap (G \setminus G^1)) + w(S \cap G^1) \leq w(S \cap (G \setminus G^1)) + \alpha_w(G^1 \setminus A_2^1)$, where again the last inequality follows from the fact that $S$ misses $A_2^1$. Now observe that $S \cap (G \setminus G^1) \cup \{u_2^1\}$ is a stable set of $G'$, and $w'(S \cap (G \setminus G^1) \cup \{u_2^1\}) = w(S \cap (G \setminus G^1)) + \tilde{w}(u_2^1)$. Then we have that $\alpha_{w'}(G') \geq w(S \cap (G \setminus G^1)) + \tilde{w}(u_2^1) \geq w(S) - \alpha_w(G^1 \setminus A_2^1) + \alpha_w(G^1 \setminus A_2^1) - \delta_1^1 = w(S) - \delta_1^1$. Now suppose that $|S \cap (A_2^1 \setminus A_1^1)| = 1$. We obtain $w(S) = w(S \cap (G \setminus G^1)) + w(S \cap G^1) \leq w(S \cap (G \setminus G^1) + \alpha_w(G^1 \setminus A_1^1)$. In this case, $S \cap (G \setminus G^1) \cup \{u_3^1\}$ is a stable set of $G'$ and this gives rise to the inequality $\alpha_{w'}(G') \geq w(S \cap (G \setminus G^1)) + \tilde{w}(u_3^1) \geq w(S) - \alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1 \setminus A_1^1) - \delta_1^1 = w(S) - \delta_1^1$. Finally, suppose that $S \cap (A_1^1 \cap A_2^1) \neq \emptyset$. Thus $w(S) = w(S \cap (G \setminus G^1)) + w(S \cap G^1) \leq w(S \cap (G \setminus G^1)) + \alpha_w(G^1)$. Moreover, $S \cap (G \setminus G^1) \cup \{u_1^1, u_3^1\}$ is a stable set of $G'$, and $w'(S \cap (G \setminus G^1) \cup \{u_1^1, u_3^1\}) = w(S \cap (G \setminus G^1)) + \tilde{w}(u_1^1) + \tilde{w}(u_3^1)$. Then we have that $\alpha_{w'}(G') \geq w(S \cap (G \setminus G^1)) + \tilde{w}(u_1^1) + \tilde{w}(u_3^1) \geq w(S) - \alpha_w(G^1) + \alpha_w(G^1) - \delta_1^1 = w(S) - \delta_1^1$.

The proof of case 3) follows the same lines of the last subcase of case 2).

Thus we have shown that for every stable set $S$ of $G$, $\alpha_{w'}(G') \geq w(S) - \delta_1^1$. In particular, this must hold for a MWSS of $G$, so we obtain $\alpha_w(G) \leq \alpha_{w'}(G') + \delta_1^1$.

Now we want to prove $\alpha_w(G) \geq \alpha_{w'}(G') + \delta_1^1$. We can partition the stable sets $S'$ of $G'$ in the following way:

1) $S' \cap \{u_1^1, u_2^1, u_3^1\} = \emptyset$;
2) $|S' \cap \{u_1^1, u_2^1, u_3^1\}| = 1$;
3) $|S' \cap \{u_1^1, u_2^1, u_3^1\}| = 2$.

In case 1), let $S^1$ be a MWSS of $G^1 \setminus (A_1^1 \cup A_2^1)$. Then, as $S'$ misses $u_1^1, u_2^1$ and $u_3^1$, and there are no edges between $G^1 \setminus (A_1^1 \cup A_2^1)$ and $G \setminus G^1$, $S^1 \cup S'$ is a stable set of $G$. It follows that $\alpha_w(G) \geq w(S' \cup S^1) = w'(S') + \delta_1^1$.

In case 2), first suppose that $u_1^1 \in S'$ and let $S^1$ be a MWSS of $G^1 \setminus A_2^1$. Then, as $S'$ misses $u_2^1$ and $u_3^1$, $S' \setminus \{u_1^1\} \cup S^1$ is a stable set of $G$, and $w(S' \setminus \{u_1^1\} \cup S^1) = w'(S') - \tilde{w}(u_1^1) + \alpha_w(G^1 \setminus A_2^1)$. It follows that $\alpha_w(G) \geq w(S' \setminus \{u_1^1\} \cup S^1) = w'(S') - \alpha_w(G^1) + \alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1 \setminus A_2^1) \geq w'(S') + \delta_1^1$, where the last inequality holds by hypothesis. Now suppose that $u_2^1 \in S'$ and let $S^1$ be a MWSS of $G^1 \setminus A_2^1$. Then, as $S'$ misses $u_1^1$ and $u_3^1$, $S' \setminus \{u_2^1\} \cup S^1$ is a stable set of $G$, and $w(S' \setminus \{u_2^1\} \cup S^1) = w'(S') - \tilde{w}(u_2^1) + \alpha_w(G^1 \setminus A_2^1)$. It follows that $\alpha_w(G) \geq w(S' \setminus \{u_2^1\} \cup S^1) = w'(S') - \alpha_w(G^1 \setminus A_2^1) + \delta_1^1 + \alpha_w(G^1 \setminus A_2^1) = w'(S') + \delta_1^1$. Finally, suppose that $u_3^1 \in S'$ and let $S^1$ be a MWSS of $G^1 \setminus A_1^1$. Then, as $S'$ misses $u_1^1$ and $u_2^1$, $S' \setminus \{u_3^1\} \cup S^1$ is a stable set of $G$, and $w(S' \setminus \{u_3^1\} \cup S^1) = w'(S') - \tilde{w}(u_3^1) + \alpha_w(G^1 \setminus A_1^1)$. It follows that $\alpha_w(G) \geq w(S' \setminus \{u_3^1\} \cup S^1) = w'(S') - \alpha_w(G^1 \setminus A_1^1) + \delta_1^1 + \alpha_w(G^1 \setminus A_1^1) = w'(S') + \delta_1^1$.

In case 3), from the structure of $T_2$, we have that $\{u_1^1, u_3^1\} \subseteq S'$. Let $S^1$ be a MWSS of $G^1$. Now, $S' \setminus \{u_1^1, u_3^1\} \cup S^1$ is a stable set of $G$, and $w(S' \setminus \{u_1^1, u_3^1\} \cup S^1) = w'(S') - \tilde{w}(u_1^1) - \tilde{w}(u_3^1) + \alpha_w(G^1)$. It follows that $\alpha_w(G) \geq w(S' \setminus \{u_1^1, u_2^1\} \cup S^1) = w'(S') - \alpha_w(G^1) + \delta_1^1 + \alpha_w(G^1) = w'(S') + \delta_1^1$.

Thus we have shown that for every stable set $S'$ of $G'$, $\alpha_w(G) \geq w'(S') + \delta_1^1$. In particular, this must hold for a MWSS of $G'$, so we obtain $\alpha_w(G) \geq \alpha_{w'}(G') + \delta_1^1$.

(c) First we prove $\alpha_w(G) \leq \alpha_{w'}(G') + \delta_1^1$. We can partition the stable sets $S$ of $G$ in the following way:

1) $S \cap (A_1^1 \cup A_2^1) = \emptyset$;
2) $|S \cap (A_1^1 \cup A_2^1)| = 1$;
3) $|S \cap (A_1^1 \cup A_2^1)| = 2$.

In case 1), we have that $w(S) = w(S \cap (G \setminus G^1)) + w(S \cap G^1) \leq w(S \cap (G \setminus G^1)) + \delta_1^1$, where the last inequality follows from the fact that $S$ misses both $A_1^1$ and $A_2^1$. Therefore, $w(S \cap (G \setminus G^1)) \geq w(S) - \delta_1^1$. Moreover, $S \cap (G \setminus G^1)$ is a stable set of $G'$ and $w'(S \cap (G \setminus G^1)) = w(S \cap (G \setminus G^1))$. It follows that $\alpha_{w'}(G') \geq w(S) - \delta_1^1$.

Now we analyze case 2). We first suppose that $S \cap (A_1^1 \setminus A_2^1) \neq \emptyset$. Then again $w(S) = w(S \cap (G \setminus G^1)) + w(S \cap G^1) \leq w(S \cap (G \setminus G^1)) + \alpha_w(G^1 \setminus A_2^1)$, where again the last inequality follows from the fact that $S$ misses $A_2^1$. Now we observe that $S \cap (G \setminus G^1) \cup \{u_1^1\}$ is a stable set of $G'$, and $w'(S \cap (G \setminus G^1) \cup \{u_1^1\}) = w(S \cap (G \setminus G^1)) + \tilde{w}(u_1^1)$. We have that $\alpha_{w'}(G') \geq w(S \cap (G \setminus G^1)) + \tilde{w}(u_1^1) \geq w(S) - \alpha_w(G^1 \setminus A_2^1) + \alpha_w(G^1 \setminus A_2^1) - \delta_1^1 = w(S) - \delta_1^1$. The case where $S \cap (A_2^1 \setminus A_1^1) \neq \emptyset$ is symmetric. Finally, suppose that $S \cap (A_1^1 \cap A_2^1) \neq \emptyset$. We have that $w(S) = w(S \cap (G \setminus G^1)) + w(S \cap G^1) \leq w(S \cap (G \setminus G^1)) + \alpha_w(G^1)$. Moreover, $S \cap (G \setminus G^1) \cup \{u_2^1\}$ is a stable set of $G'$, and $w'(S \cap (G \setminus G^1) \cup \{u_2^1\}) = w(S \cap (G \setminus G^1)) + \tilde{w}(u_2^1)$. Then we have that $\alpha_{w'}(G') \geq w(S \cap (G \setminus G^1)) + \tilde{w}(u_2^1) \geq w(S) - \alpha_w(G^1) + \alpha_w(G^1) - \delta_1^1 = w(S) - \delta_1^1$.

The proof of case 3) goes along the same lines of the last subcase of case 2).

Thus we have shown that for every stable set $S$ of $G$, $\alpha_{w'}(G') \geq w(S) - \delta_1^1$. In particular, this must hold for a MWSS of $G$, so we obtain $\alpha_w(G) \leq \alpha_{w'}(G') + \delta_1^1$.

Now we want to prove $\alpha_w(G) \geq \alpha_{w'}(G') + \delta_1^1$. We can partition the stable sets $S'$ of $G'$ in the following way:

1) $S' \cap \{u_1^1, u_2^1, u_3^1\} = \emptyset$;
2) $|S' \cap \{u_1^1, u_2^1, u_3^1\}| = 1$;
3) $|S' \cap \{u_1^1, u_2^1, u_3^1\}| = 2$.

In case 1), let $S^1$ be a MWSS of $G^1 \setminus (A_1^1 \cup A_2^1)$. Then, as $S'$ misses $u_1^1, u_2^1$ and $u_3^1$, and there are no edges between $G^1 \setminus (A_1^1 \cup A_2^1)$ and $G \setminus G^1$, $S^1 \cup S'$ is a stable set of $G$. It follows that $\alpha_w(G) \geq w(S' \cup S^1) = w'(S') + \delta_1^1$.

In case 2), first suppose that $u_1^1 \in S'$ and let $S^1$ be a MWSS of $G^1 \setminus A_2^1$. Then, as $S'$ misses $u_2^1$ and $u_3^1$, $S' \setminus \{u_1^1\} \cup S^1$ is a stable set of $G$, and $w(S' \setminus \{u_1^1\} \cup S^1) = w'(S') - \tilde{w}(u_1^1) + \alpha_w(G^1 \setminus A_2^1)$. It follows that $\alpha_w(G) \geq w(S' \setminus \{u_1^1\} \cup S^1) = w'(S') - \alpha_w(G^1 \setminus A_2^1) + \delta_1^1 + \alpha_w(G^1 \setminus A_2^1) = w'(S') + \delta_1^1$. The case where $u_3^1 \in S'$ goes along the same lines. Finally, let us suppose that $u_2^1 \in S'$ and let $S^1$ be a MWSS of $G^1$. Then $S' \setminus \{u_2^1\} \cup S^1$ is a stable set of $G$, and $w(S' \setminus \{u_2^1\} \cup S^1) = w'(S') - \tilde{w}(u_2^1) + \alpha_w(G^1)$. It follows that $\alpha_w(G) \geq w(S' \setminus \{u_2^1\} \cup S^1) = w'(S') - \alpha_w(G^1) + \delta_1^1 + \alpha_w(G^1) = w'(S') + \delta_1^1$.

In case 3), from the structure of $T_2$, we have that $\{u_1^1, u_3^1\} \subseteq S'$. Let $S^1$ be a MWSS of $G^1$. Now, $S' \setminus \{u_1^1, u_3^1\} \cup S^1$ is a stable set of $G$, and $w(S' \setminus \{u_1^1, u_3^1\} \cup S^1) = w'(S') - \tilde{w}(u_1^1) - \tilde{w}(u_3^1) + \alpha_w(G^1)$. It follows that $\alpha_w(G) \geq w(S' \setminus \{u_1^1, u_2^1\} \cup S^1) = w'(S') - \alpha_w(G^1 \setminus A_2^1) + 2\delta_1^1 - \alpha_w(G^1 \setminus A_1^1) + \alpha_w(G^1) \geq w'(S') + \delta_1^1$, where the last inequality holds by hypothesis.

Thus we have shown that for every stable set $S'$ of $G'$, $\alpha_w(G) \geq w'(S') + \delta_1^1$. In particular, this must hold for a MWSS of $G'$, so we obtain $\alpha_w(G) \geq \alpha_{w'}(G') + \delta_1^1$.

□

# C Example.

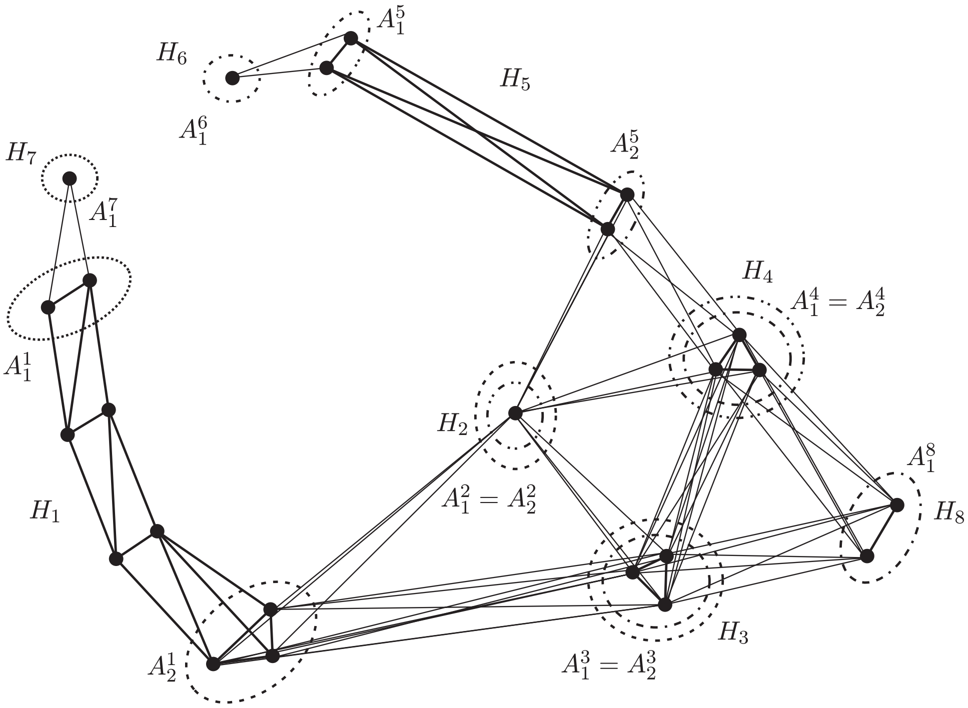


Figure 2: A graph $G$, composition of the 2-strips $H_1, \ldots, H_5$ and the 1-strips $H_6, H_7, H_8$. Partition $\mathcal{P}$ is given by
$\{\{A_1^1, A_1^7\}, \{A_1^5, A_1^6\}, \{A_2^1, A_2^2, A_2^3\}, \{A_1^2, A_2^5, A_1^4\}, \{A_1^3, A_2^4, A_1^8\}\}$.

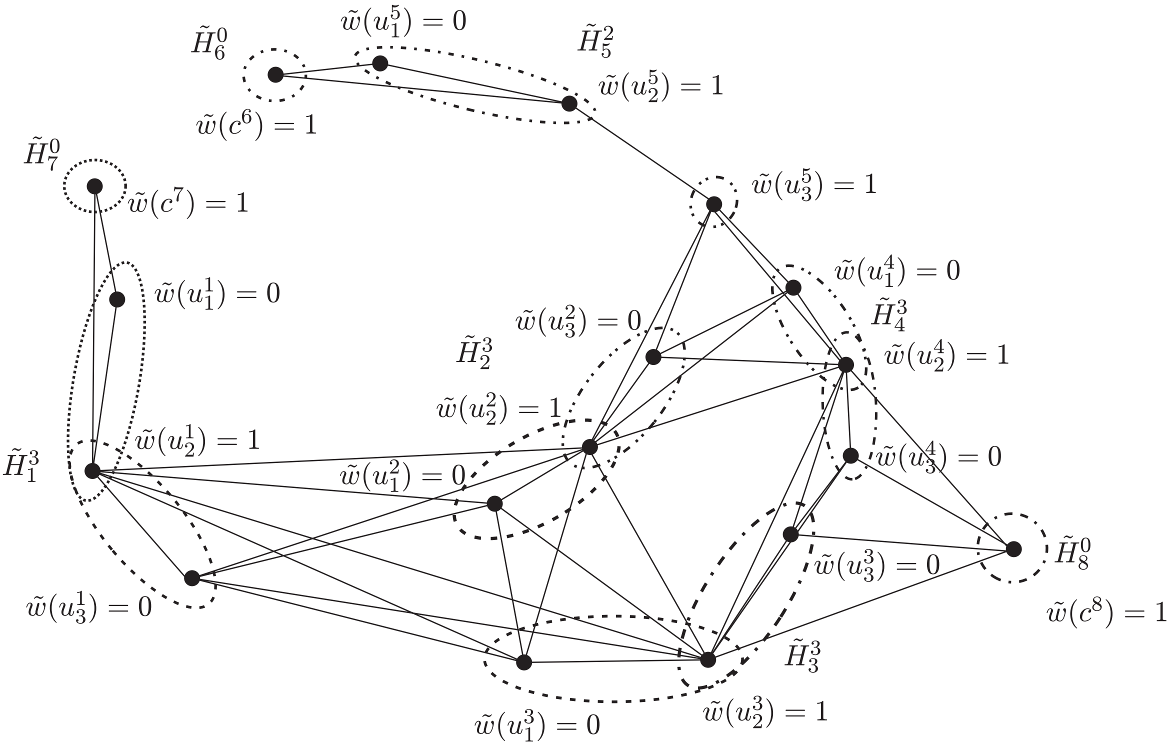


Figure 3: The weighted graph $\tilde{G}$, corresponding to graph $G$ in Figure 2, and the weight function $w$ such that $w(v) = 1$ for every vertex $v$ of $G$.

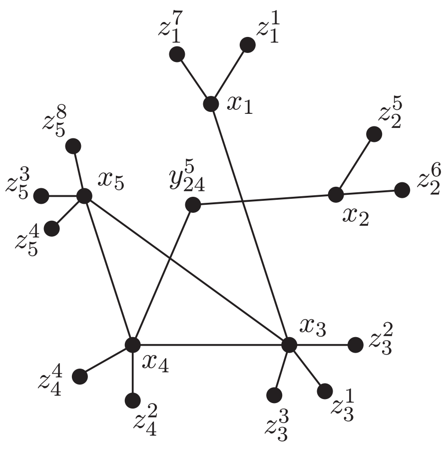


Figure 4: The root graph $H$ of the line graph $\tilde{G}$ in Figure 3.