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Partial characterizations of clique-perfect and coordinated graphs: superclasses of triangle-free graphs

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Abstract

A graph G is *clique-perfect* if the cardinality of a maximum clique-independent set of H equals the cardinality of a minimum clique-transversal of H, for every induced subgraph H of G. A graph G is *coordinated* if the minimum number of colors that can be assigned to the cliques of H in such a way that no two cliques with non-empty intersection receive the same color equals the maximum number of cliques of H with a common vertex, for every induced subgraph H of G. Coordinated graphs are a subclass of perfect graphs. The complete lists of minimal forbidden induced subgraphs for the classes of clique-perfect and coordinated graphs are not known, but some partial characterizations have been obtained. In this paper, we characterize clique-perfect and coordinated graphs by minimal forbidden induced subgraphs when the graph is either paw-free or {gem,W_4,bull}-free, both superclasses of triangle-free graphs.

Key words: Clique-perfect graphs, coordinated graphs, {gem,W_4,bull}-free graphs, paw-free graphs, perfect graphs, triangle-free graphs.

1 Introduction

Let G be a simple finite undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$ the complement of G. A graph with only one vertex will be called a *trivial* graph. Given two graphs G and G', we say that G *contains* G' if G' is isomorphic to an induced subgraph of G. When we need to refer to the non-induced subgraph containment relation, we will mention it explicitly.

A *complete set* or just a *complete* of a graph is a subset of pairwise adjacent vertices. A complete composed by three vertices is called a *triangle*. A *clique* is a complete set not properly contained in any other complete set. We may also use the term clique to refer to the corresponding complete subgraph. Given a graph G and a vertex v in $V(G)$, we denote by $m(v)$ the number of cliques including the vertex v.

A *stable set* in a graph G is a subset of pairwise non-adjacent vertices of G. A graph is *bipartite* if its vertex set can be partitioned into two stable sets.

Let X and Y be two sets of vertices of G. We say that X is *complete to* Y if every vertex in X is adjacent to every vertex in Y, and that X is *anticomplete to* Y if no vertex of X is adjacent to a vertex of Y.

A vertex v of a graph G is called *universal* if it is adjacent to every other vertex of G, and it is called a *leaf* of G if it has degree one in G.

We say that a graph G is *anticonnected* if $\overline{G}$ is connected. An *anticomponent* of a graph G is a connected component of $\overline{G}$. A graph is called *complete multipartite* if it is not anticonnected and all its anticomponents are stable sets.

A *hole* is a chordless cycle of length at least 4. An *antihole* is the complement of a hole. A hole or antihole is said to be *odd* if it has an odd number of vertices. A hole of length j is denoted by C_j. Denote by P_j the induced path of j vertices.

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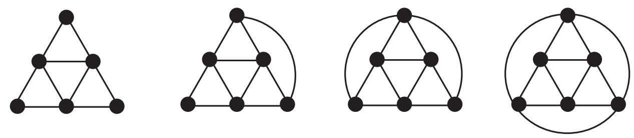

Fig. 1. Forbidden induced subgraphs for the class of HCH graphs.

A *gem* is a graph of five vertices, such that four of them induce a P_4 and the fifth vertex is universal. A *wheel* W_j is a graph of $j+1$ vertices, such that j of them induce a C_j and the last vertex is universal. A *paw* is a triangle with a leaf attached to one of its vertices. A *bull* is a triangle with two leafs attached to different vertices of it.

The *chromatic number* of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and it is denoted by $\chi(G)$. An obvious lower bound of $\chi(G)$ is the maximum cardinality of a clique in G, the *clique number* of G, denoted by $\omega(G)$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G [1]. Complete graphs, bipartite graphs, and line graphs of bipartite graphs are perfect [9]. In [17] it was proved that a graph is perfect if and only if its complement is perfect. The characterization of perfect graphs by forbidden induced subgraphs has been proved recently: a graph G is perfect if and only if no induced subgraph of G is an odd hole or an odd antihole [7]. Besides, perfect graphs can be recognized in polynomial time [6].

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being adjacent if and only if the corresponding sets have nonempty intersection.

The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. A graph G is *K-perfect* if $K(G)$ is perfect.

A family S of sets is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element. A graph G is *clique-Helly* (CH) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (HCH) if H is clique-Helly for every induced subgraph H of G. A graph G is HCH if and only if G does not contain any of the graphs in Figure 1 as an induced subgraph [21].

A *clique-transversal* of a graph G is a subset of vertices meeting all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_C(G)$ and $\alpha_C(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. Clearly, $\alpha_C(G) \leq \tau_C(G)$ for any graph G. A graph G is *clique-perfect* if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G. Clique-perfect graphs have been implicitly studied in several works but the term "clique-perfect" has been introduced in [10]. The only clique-perfect graphs which are minimally imperfect are $\overline{C_{6j+3}}$, for any $j \geq 1$ [8].

A *K-coloring* of a graph G is an assignment of colors to the cliques of G in such a way that no two cliques with non-empty intersection receive the same color (equivalently, a K-coloring of G is a coloring of $K(G)$). A *Helly K-complete* of a graph G is a collection of cliques of G with common intersection. The *K-chromatic number* and *Helly K-clique number* of G, denoted by $F(G)$ and $M(G)$, are the sizes of a minimum K-coloring and a maximum Helly K-complete of G, respectively. It is easy to verify that $F(G) = \chi(K(G))$ and that $M(G) = \max_{v \in V(G)} m(v)$. Also, $F(G) \geq M(G)$ for any graph G. A graph G is *C-good* if $F(G) = M(G)$. A graph G is *coordinated* if every induced subgraph of G is C-good. Coordinated graphs were defined and studied in [4], where it was proved that they are a subclass of perfect graphs.

The recognition problem for coordinated graphs is NP-hard. Furthermore, this problem is NP-complete when restricted to {gem,W_4,C_4}-free graphs G with $M(G) \leq 3$ [23]. The complexity of the recognition problem for clique-perfect graphs is still unknown.

Bipartite graphs are clique-perfect and coordinated [13,14].

A class of graphs $\mathcal{C}$ is *hereditary* if for every $G \in \mathcal{C}$, every induced subgraph of G also belongs to $\mathcal{C}$. If $\mathcal{C}$ is a hereditary class of K-perfect clique-Helly graphs, then every graph in $\mathcal{C}$ is clique-perfect and coordinated [2,5].

Finding the complete lists of minimal forbidden induced subgraphs for the classes of clique-perfect and coordinated graphs turns out to be a difficult task [2,24]. However, some partial characterizations have been obtained in previous works. In [16], clique-perfect graphs are characterized by minimal forbidden subgraphs for the class of chordal graphs. In [2] and [3], clique-perfect graphs are characterized by minimal forbidden subgraphs for two subclasses of claw-free graphs, and for Helly circular-arc graphs, respectively. In the same direction, coordinated graphs are characterized by minimal forbidden subgraphs for line graphs and complements of forests [5].

In this paper, we characterize clique-perfect and coordinated graphs by minimal forbidden induced subgraphs when the graph lies in one of two superclasses of triangle-free graphs: paw-free and {gem,W_4,bull}-free graphs. In particular, we prove that in these cases both classes are equivalent to perfect graphs and, in consequence, the only forbidden subgraphs are the odd holes (odd antiholes of length at least seven are neither paw-free nor {gem,W_4,bull}-free). As a direct corollary, we can deduce polynomial-time algorithms to recognize clique-perfect and coordinated graphs when the graph belongs to these classes.

2 Superclasses of triangle-free graphs

A graph is triangle-free if it contains no triangle as induced subgraph. Triangle-free graphs were extensively studied in the literature, usually in the context of graph coloring problems (see for example [12,18,19]).

It is interesting to remark that if the graph G is triangle-free, then $F(G)$ equals the chromatic index of G and $M(G)$ equals the maximum degree of G. Hence, the graph G is coordinated if and only if every induced subgraph H of G belongs to Class 1 (i.e., graphs where the chromatic index equals the maximum degree).

It is easy to see that if a graph G is triangle-free, then G is perfect if and only if G is clique-perfect, if and only if G is coordinated. In order to prove this, we only need to use the following facts: odd holes are neither perfect, nor clique-perfect, nor coordinated; graphs with neither triangles nor odd holes are bipartite; and bipartite graphs are perfect, clique-perfect and coordinated. Therefore, it is enough to forbid odd holes to characterize clique-perfect (and coordinated) graphs in this case. We shall extend this result by analyzing two superclasses of triangle-free graphs: paw-free and {gem,W_4,bull}-free graphs.

2.1 Paw-free graphs

A graph is paw-free if it contains no paw as induced subgraph. Paw-free graphs were studied in [20]. This class is interesting to analyze because it contains graphs with an exponential number of cliques, while in most of the classes where a forbidden subgraph characterization or a polynomial-time recognition algorithm for clique-perfect or coordinated graphs is known, the number of cliques is polynomially bounded (e.g., chordal graphs, diamond-free graphs, claw-free HCH graphs, Helly circular-arc graphs, and line graphs).

In this section we prove that the characterization mentioned above for clique-perfect and coordinated graphs on triangle-free graphs also holds for paw-free graphs.

The proof of this result can be divided into two cases: the case when G is anticonnected and the case when G is not anticonnected.

In the first case, we shall resort to the following result presented in [20]: if G is also connected, then G contains no triangles (Lemma 2). Furthermore, it is shown that if G is anticonnected, then G is perfect if and only if G is bipartite (Corollary 4), and bipartite graphs are clique-perfect and coordinated. Finally, if G is clique-perfect and does not contain triangles, then G is perfect.

In the second case, we shall rely on the fact that all the anticomponents of G are stable sets (Lemma 1), so an appropriate coloring of $K(G)$ for this kind of graphs is found (Theorem 5) for the coordinated case, and the clique-perfectness follows from known results.

Lemma 1 [20] *Let G be a paw-free not anticonnected graph. Then the anticomponents of G are stable sets, i.e., G is a complete multipartite graph.*

Lemma 2 [20] *Let G be a paw-free connected and anticonnected graph. Then G is triangle-free.*

We first prove the following auxiliary results.

Proposition 3 *Let G be a connected graph. Then the following statements are equivalent:*

(i) *G is perfect, paw-free, and it has at most two anticomponents.*
(ii) *G is bipartite.*

PROOF.
(i) $\Rightarrow$ (ii)) If G is not anticonnected, then by Lemma 1 the anticomponents of G are stable sets. The graph G has at most two anticomponents, so it is bipartite.

If G is anticonnected, since G is connected and paw-free, then G is triangle-free by Lemma 2. As G is also perfect, it does not have odd holes. If G contains no triangles and contains no odd holes, then G contains no odd cycles as subgraphs. Therefore, G is bipartite.

(ii) $\Rightarrow$ (i)) Trivial. $\square$

We have, therefore, the following straightforward corollary.

Corollary 4 *Let G be a paw-free, connected, and anticonnected graph. Then G is perfect if and only if G is bipartite.*

Complete multipartite graphs are a subclass of distance-hereditary graphs. In [15] it is proved that distance-hereditary graphs are clique-perfect, hence complete multipartite graphs are clique-perfect.

Theorem 5 *If G is a complete multipartite graph, then G is coordinated.*

PROOF. Complete multipartite graphs are clearly hereditary. Then, it is enough to see that every graph in this class is C-good.

Let H be a complete multipartite graph. Let $A_1, \ldots, A_k$ ($k \geq 1$) be the anti-components of H. We can assume that $|A_i| \leq |A_{i+1}|$ ($1 \leq i < k$).

Let $b = |A_k|$, i.e., the size of the biggest anticomponent of H. If $b = 1$, then H is complete and is, therefore, trivially C-good. We thus assume $b > 1$.

Every clique of H has exactly one vertex in each anticomponent, hence $m(v) = \prod_{i=1, i\neq j}^{i=k} |A_i|$ for each vertex $v \in A_j$. Since A_1 is the smallest anticomponent, $M(H) = \prod_{i=2}^{i=k} |A_i|$.

Furthermore, there is a one-to-one correspondence between the cliques of H and the sequences $[a_1, \ldots, a_k]$ with $0 \leq a_i \leq |A_i| - 1$. Let $\mathcal{A}$ be the set of all such sequences, and let $c : \mathcal{A} \to \mathbb{N}_0$ be defined as follows:

$$c(0, a_2, \ldots, a_k) = \sum_{i=2}^{k} a_i b^{i-2}, \tag{1}$$

$$c(a_1, a_2, \ldots, a_k) = c(0, r(a_2 - a_1, |A_2|), \ldots, r(a_k - a_1, |A_k|)) \text{ if } a_1 > 0, \tag{2}$$

where $r(x, z)$ denotes the remainder of the integer division x/z. We shall use c as a coloring of the cliques of H.

The number of sequences in $\mathcal{A}$ with $a_0 = 0$ is $\prod_{i=2}^{i=k} |A_i|$, so the function c uses at most $M(H)$ colors. If c is a valid coloring then $M(H) = F(H)$, implying that H is C-good.

We now check that c is a valid coloring. Consider two sequences $a = [a_1, \ldots, a_k]$, $a' = [a'_1, \ldots, a'_k] \in \mathcal{A}$, such that $c(a) = c(a')$. We shall prove that either $a = a'$ or a does not intersect a' (that is, $a_i \neq a'_i$ for all $1 \leq i \leq k$).

By (2) and (1), we get

$$c(a) = c(0, r(a_2 - a_1, |A_2|), \ldots, r(a_k - a_1, |A_k|)) = \sum_{i=2}^{k} r(a_i - a_1, |A_i|) b^{i-2}$$

and, similarly,

$$c(a') = \sum_{i=2}^{k} r(a'_i - a'_1, |A_i|) b^{i-2}.$$

Since $c(a) = c(a')$, we have

$$\sum_{i=2}^{k} r(a_i - a_1, |A_i|) b^{i-2} = \sum_{i=2}^{k} r(a'_i - a'_1, |A_i|) b^{i-2}.$$

Since $b > 1$ and $0 \leq r(a_i - a_1, |A_i|), r(a'_i - a'_1, |A_i|) < |A_i| \leq b$. By the uniqueness of representation of a natural number in base b, it follows that $r(a_i - a_1, |A_i|) = r(a'_i - a'_1, |A_i|)$ for all $2 \leq i \leq k$. That is, $a_i - a_1 \equiv a'_i - a'_1 \mod |A_i|$ for all $2 \leq i \leq k$.

Therefore, for each $2 \leq i \leq k$, $a_1 \equiv a'_1 \mod |A_i|$ if and only if $a_i \equiv a'_i \mod |A_i|$. But, since $0 \leq a_i, a'_i < |A_i|$ and $0 \leq a_1, a'_1 < |A_1| \leq |A_i|$, it follows that $a_1 = a'_1$ if and only if $a_1 \equiv a'_1 \mod |A_i|$, if and only if $a_i \equiv a'_i \mod |A_i|$, if and only if $a_i = a'_i$. So, if $a_1 = a'_1$ then $a_i = a'_i$ for every $2 \leq i \leq k$, and if $a_1 \neq a'_1$ then $a_i \neq a'_i$ for every $2 \leq i \leq k$. That is, either $a = a'$ or the cliques corresponding to a and a' do not intersect. $\square$

We are now in position of proving the main result of this section.

Theorem 6 *If G is a paw-free graph, then the following statements are equivalent:*

(i) *G is perfect.*
(ii) *G is clique-perfect.*
(iii) *G is coordinated.*

PROOF.

(i) $\Rightarrow$ (ii)) If G is not anticonnected, then by Lemma 1 G is a complete multipartite graph, so G is clique-perfect [15]. Otherwise, without loss of generality, we can assume that G is connected. Then, by Corollary 4, G is bipartite and so G is clique-perfect.

(ii) $\Rightarrow$ (iii)) If G is not anticonnected, then by Lemma 1 and Theorem 5, G is coordinated. Otherwise, without loss of generality, we can assume that G is connected. By Lemma 2, G has no triangles and therefore G does not have odd antiholes with length greater than 5. On the other hand, as odd holes are not clique-perfect, G has no odd holes. We conclude that G is perfect. Let G' be an induced subgraph of G. To see that G' is C-good, it is enough to prove that every connected component of G' is C-good. Let H be a connected component of G'. If H is not anticonnected, then H is coordinated, by Lemma 1 and Theorem 5; in particular it is C-good. If H is anticonnected, since it is also connected and perfect, it follows by Corollary 4 that H is bipartite. Then H is C-good.

(iii) ⇒ (i)) Coordinated graphs are a subclass of perfect graphs. □

As a consequence of these results, the recognition problem can be solved in linear time.

Theorem 7 *The problem of determining if a paw-free graph is clique-perfect (coordinated) can be solved in linear time.*

PROOF. Check every connected component of the graph looking for one component that is anticonnected and not bipartite. If such a component exists, then return "the graph is not clique-perfect (coordinated)". Otherwise, return "the graph is clique-perfect (coordinated)".

This algorithm clearly runs in linear time with respect to the size of the input. The correctness is a consequence of Corollary 4 and Theorems 5 and 6. □

2.2 Another superclass of triangle-free graphs: $\{gem, W_4, bull\}$-free graphs

Bull-free graphs have been studied in the context of perfect graphs [11,22], and $\{\text{gem}, W_4\}$-free graphs have been considered in the context of clique-perfect graphs [8]. Recall that the recognition of coordinated graphs is NP-Hard in $\{\text{gem}, W_4, C_4\}$-free graphs [23].

We analyze here another superclass of triangle-free graphs: $\{\text{gem}, W_4, \text{bull}\}$-free graphs. We prove that if such a graph is perfect, then it is K-perfect. By the forbidden subgraph characterization of HCH graphs, $\{\text{gem}, W_4\}$-free graphs are also HCH. Since the class of $\{\text{gem}, W_4, \text{bull}\}$-free graphs is hereditary, we obtain as a corollary ([2,5]) that $\{\text{gem}, W_4, \text{bull}\}$-free graphs are clique-perfect (coordinated) if and only if they are also perfect, the same result that holds for triangle-free graphs.

It is interesting to remark that this result does not hold for $\{\text{gem}, W_4\}$-free graphs. It is not difficult to build examples of $\{\text{gem}, W_4\}$-free perfect graphs which are neither clique-perfect nor coordinated.

In order to show that a perfect $\{\text{gem}, W_4, \text{bull}\}$-free graph G is K-perfect, we need to prove that $K(G)$ contains neither odd holes nor odd antiholes. We begin by proving that no induced subgraph of $K(G)$ is an odd antihole of length at least 7.

Theorem 8 *If G is a $\{gem, W_4\}$-free graph then $K(G)$ is a $\{gem, W_4\}$-free graph.*

PROOF. Suppose that there exist cliques $Q_1, \ldots, Q_4$ of G such that $Q'_1, \ldots, Q'_4$ (the corresponding vertices in $K(G)$) induce a path or hole in $K(G)$ (in that order), and let Q_0 be a clique having common intersection with all of $Q_1, \ldots, Q_4$. Define $V_2 = (Q_0 \cap Q_1 \cap Q_2)$ and $V_3 = (Q_0 \cap Q_3 \cap Q_4)$, which are non-empty because G is HCH, and choose $v_2 \in V_2$ and $v_3 \in V_3$. From $Q_2 \cap Q_4 = \emptyset$, we obtain $Q_2 \cap V_3 = \emptyset$. Consequently, there exists a vertex $v_1 \in Q_2$ which is non-adjacent to v_3. In a similar way, there exists a vertex $v_4 \in Q_3$ which is non-adjacent to v_2.

Both v_2 and v_1 belong to Q_2, so they are adjacent. Similarly, v_3 and v_4 are also adjacent because they both belong to Q_3. Finally, v_2 and v_3 are adjacent because they both belong to Q_0. Therefore, v_1, v_2, v_3, v_4 induce a path or a hole in G. Choose $v_0 \in Q_2 \cap Q_3$. Then v_0 is adjacent (and different) to all of v_1, v_2, v_3, v_4, so v_0, v_1, v_2, v_3, v_4 induce a gem or W_4 in G, which is a contradiction. □

Any antihole of length at least seven contains a gem, thus we have the following corollary.

Corollary 9 *If G is a {gem,W_4}-free graph then $K(G)$ contains no odd antihole of length greater than 5.*

Let G be a graph. A *hole of cliques* $Q_1, \ldots, Q_k$ ($k \geq 4$) is a set of cliques of G which induces a hole in $K(G)$ (i.e., $Q_i \cap Q_j \neq \emptyset \Leftrightarrow i = j$ or $i \equiv j \pm 1 \mod k$). An *intersection cycle of a hole of cliques* $Q_1, \ldots, Q_k$ is a cycle $v_1, \ldots, v_k$ of G such that $v_i \in Q_i \cap Q_{i+1}$ for every $i = 1, \ldots, k$. Let $C = v_1, \ldots, v_k$ be an intersection cycle of a hole of cliques $Q_1, \ldots, Q_k$. The clique Q_{i+1} will be denoted either by $Q_C(v_i, v_{i+1})$ or by $Q_C(v_{i+1}, v_i)$. When the cycle C is clear from the context, we note simply $Q(v_i, v_{i+1})$ or $Q(v_{i+1}, v_i)$.

We proceed to prove that if G is perfect and {gem,W_4,bull}-free, then $K(G)$ has no induced odd hole. To this end, we introduce the following lemmas, some of which are trivial and stated with no proof.

Lemma 10 *Let G be a {gem,W_4}-free graph and $C = v_1, \ldots, v_{2k+1}$ ($k \geq 2$) an intersection cycle of a hole of cliques of G. Then*

(1) C has no short chord, and
(2) no vertex of C is adjacent to three consecutive vertices of C.

PROOF.
(1) If v_{i-1} is adjacent to v_{i+1}, since $Q(v_{i-1}, v_i)$ is a clique and $v_{i+1} \notin Q(v_{i-1}, v_i)$, there exists a vertex $w_{i-1} \in Q(v_{i-1}, v_i)$ non-adjacent to v_{i+1}. In a similar way,

there exists another vertex $w_{i+1} \in Q(v_{i+1}, v_i)$ non-adjacent to v_{i-1}. Therefore $v_i, w_{i-1}, v_{i-1}, v_{i+1}, w_{i+1}$ induce a gem or a W_4.

(2) If v_i is adjacent to three consecutive vertices v_j, v_{j+1}, v_{j+2}, since $Q(v_j, v_{j+1})$ is a clique, there exists a vertex $w \in Q(v_j, v_{j+1})$ which is not adjacent to v_i. On the other hand, by item 1, v_j is not adjacent to v_{j+2}. Therefore $v_{j+1}, w, v_j, v_i, v_{j+2}$ induce a gem or a W_4. $\square$

The next two lemmas are straightforward.

Lemma 11 *Let G be a $\{gem, W_4\}$-free graph, $C = v_1, \ldots, v_{2k+1}$ $(k \geq 2)$ be an intersection cycle of a hole of cliques of G, v_i, v_j, v_l be a triangle, and $d \in \{-1, 1\}$. If $i + d \neq j$ and $i + d \neq l$, then v_j and v_l are both adjacent to v_{i+d} or both non-adjacent to v_{i+d}.*

Lemma 12 *Let G be a bull-free graph, and $C = v_1, \ldots, v_{2k+1}$ $(k \geq 2)$ be a cycle and let $i', j', l' \in \{-1, 1\}$. If v_i, v_j, v_l induce a triangle, $v_{i+i'}$ is adjacent to neither v_j nor v_l, $v_{j+j'}$ is adjacent to neither v_i nor v_l, and $v_{l+l'}$ is adjacent to neither v_i nor v_j, then $v_{i+i'}, v_{j+j'}, v_{l+l'}$ induce a triangle.*

Lemma 13 *Let G be a $\{gem, W_4, bull\}$-free graph, $C = v_1, \ldots, v_{2k+1}$ be an intersection cycle of a hole of cliques of G and $d \in \{1, -1\}$. If v_i, v_j, v_{j+1} induce a triangle, then v_{i+d}, v_j, v_{j+1} induce a triangle, or $v_{i+d}, v_{j-1}, v_{j+2}$ induce a triangle.*

PROOF. By item (1) of Lemma 10, v_{j-1} is non-adjacent to v_{j+1} and v_j is not adjacent to v_{j+2}. In particular, $i+d$ differs from j and $j+1$. The vertex v_i is adjacent to both v_j and v_{j+1}, therefore, item (2) of Lemma 10 implies that v_i is adjacent to neither v_{j-1} nor v_{j+2}.

Suppose that v_{i+d}, v_j, v_{j+1} is not a triangle. By Lemma 11, v_{i+d} is adjacent to neither v_j nor v_{j+1}. Then, v_i, v_j, v_{j+1} induce a triangle, v_{i+d} is adjacent to neither v_j nor v_{j+1}; v_{j-1} is adjacent to neither v_i nor v_{j+1}; v_{j+2} is adjacent to neither v_i nor v_j. Thus, by Lemma 12, $v_{i+d}, v_{j-1}, v_{j+2}$ induce a triangle. $\square$

Lemma 14 *Let G be a $\{gem, W_4, bull\}$-free graph, $C = v_1, \ldots, v_{2k+1}$ $(k \geq 2)$ be an intersection cycle of a hole of cliques of G, v_i, v_{j-1}, v_{j+2} be a triangle, and $d \in \{-1, 1\}$. If $i + d \neq j - 1$ and $i + d \neq j + 2$, then $v_{i+d}, v_{j-1}, v_{j+2}$ or v_{i+d}, v_j, v_{j+1} induce a triangle.*

PROOF. By item (1) of Lemma 10, C has no short chord. In particular, i differs from j and $j+1$; v_j is non-adjacent to v_{j+2} and v_{j-1} is non-adjacent to v_{j+1}. By Lemma 11 (with $i := j-1$, $i+d := j$, $j := i$, $l := j+2$, recalling that

v_i, v_{j-1}, v_{j+2} is a triangle), v_j is non-adjacent to v_i. Using the same argument, we obtain that v_{j+1} is non-adjacent to v_i.

Suppose that $v_{i+d}, v_{j-1}, v_{j+2}$ is not a triangle. By Lemma 11, v_{i+d} is adjacent to neither v_{j-1} nor v_{j+2}. Therefore, v_i, v_{j-1}, v_{j+2} induce a triangle; v_{i+d} is adjacent to neither v_{j-1} nor v_{j+2}; v_j is adjacent to neither v_i nor v_{j+2}; v_{j+1} is adjacent to neither v_i nor v_{j-1}. Hence, Lemma 12 implies that v_{i+d}, v_j, v_{j+1} induce a triangle. □

Let C be a cycle of a graph G. An edge (v, w) of C is *improper* if there is a vertex $z \in C$ such that v, w, z is a triangle. Conversely, an edge of C is *proper* if it is not improper. A vertex of C is *lonely* if it does not induce a triangle with any two other vertices of C.

In order to prove our main theorem we are going to show that if (v_i, v_{i+1}) is an improper edge of an intersection cycle $v_1, \ldots, v_{2k+1}$ $(k \geq 2)$ of a hole of cliques of G, then (v_{i+1}, v_{i+2}) is a proper edge. Also, if (v_i, v_{i+1}) is a proper edge then (v_{i+1}, v_{i+2}) is an improper edge. Therefore, there is no such odd-length intersection cycle.

Lemma 15 *Let G be a perfect $\{gem, W_4, bull\}$-free graph and $C = v_1, \ldots, v_{2k+1}$ $(k \geq 2)$ be an intersection cycle of a hole of cliques of G. Then no vertex of C is lonely.*

PROOF. By contradiction, suppose that C contains lonely vertices. Since G is perfect and C is an odd cycle, it follows that C must have three vertices inducing a triangle. Therefore, we can find a lonely vertex v_i such that v_{i+1} is not lonely. Let v_j, v_{j+l} be two vertices such that v_{i+1}, v_j, v_{j+l} induce a triangle. Without loss of generality, we may assume that $i + 1 < j < j + l$ and that j and l are chosen so that l is minimum. Since v_i is lonely, it follows that $i \neq j$ and $i \neq j + l$.

If $l = 1$ (i.e., v_{i+1}, v_j, v_{j+1} is a triangle) then by Lemma 13 (taking $i := i + 1$) it follows that v_i, v_j, v_{j+1} induce a triangle or v_i, v_{j-1}, v_{j+2} induce a triangle, contradicting the fact that v_i is lonely. By item (1) of Lemma 10, C has no short chord, so v_j is not adjacent to v_{j+2}. Therefore, $l \geq 3$.

From $l \geq 3$ we obtain $i + 1 < j + 1 < j + l$ and, in particular, v_{i+1}, v_{j+1} and v_{j+l} are three different vertices. Moreover, since we chose j and l such that l is minimum, v_{j+1} is non-adjacent either to v_{j+l} or to v_{i+1} (otherwise, we may choose v_{j+1} instead of v_j). By Lemma 11 (taking $i := j$, $j := i + 1$, $l := j + l$), it follows that both v_{j+l} and v_{i+1} are non-adjacent to v_{j+1}. By the same argument, interchanging $j+1$ with $j+l-1$ and $j+l$ with j, we conclude that v_{j+l-1} is adjacent to neither v_j nor v_{i+1}.

We have that v_{i+1}, v_j, v_{j+l} induce a triangle; v_i is adjacent to neither v_j nor v_{j+l}; v_{j+1} is adjacent to neither v_{j+l} nor v_{i+1}; v_{j+l-1} is adjacent to neither v_j nor v_{i+1}. By Lemma 12, v_i, v_{j+l-1}, v_{j+1} induce a triangle, contradicting the fact that v_i is lonely. □

Lemma 16 *Let G be a perfect $\{gem, W_4, bull\}$-free graph and $C = v_1, \ldots, v_{2k+1}$ $(k \geq 2)$ be an intersection cycle of a hole of cliques of G. Then C does not contain two consecutive improper edges.*

PROOF. Suppose the lemma is false. Then, there are vertices v_{i-1}, v_i, v_{i+1} such that v_{i-1}, v_i, v_j is a triangle and v_i, v_{i+1}, v_{j+h} is another triangle. Let $I = \{v_j,\ v_{j+sg(h)},\ \ldots,\ v_{j+h}\}$ (where $sg(h) = 1$ if $h > 0$,-1 if $h < 0$ and 0 if $h = 0$). We can choose h to be positive or negative, so that none of v_{i-1}, v_i, v_{i+1} belongs to I. We may also assume that j and h are taken such that $|h|$ is minimum satisfying these conditions. For ease of notation, call $w_j = v_j$ and $w_{j+s} = v_{j+s \times sg(h)}$ for all $1 \leq s \leq |h|$. Also call $l = |h|$.

By item (2) of Lemma 10, w_j is non-adjacent to v_{i+1} because w_j is adjacent to both v_{i-1} and v_i. Similarly, w_{j+l} is non-adjacent to v_{i-1}. Then $w_{j+l} \neq w_j$, so $l > 0$.

By item (1) of Lemma 10, C has no short chord and therefore v_{i-1} is non-adjacent to v_{i+1}. If $l = 1$ then $v_i, v_{i-1}, w_j, w_{j+1}, v_{i+1}$ induce a gem, which is a contradiction, so $l \geq 2$. From $l \geq 2$, v_{i-1}, v_i, w_{j+1} is not a triangle, otherwise we could choose w_{j+1} instead of w_j contradicting the minimality of $l = |h|$. Clearly, $w_{j+1} \in I$ and $v_i, v_{i-1} \notin I$, so they are all different. By Lemma 13, $w_{j+1}, v_{i-2}, v_{i+1}$ induce a triangle.

Suppose $l = 2$. Then $w_{j+l} = w_{j+2}$ is adjacent to v_{i+1}. The vertex w_{j+1} is also adjacent to v_{i+1}, $v_i \neq w_{j+2}$, $v_i \neq w_{j+1}$, and v_i is adjacent to w_{j+2}. Therefore, Lemma 11 implies that v_i is also adjacent to w_{j+1}. We have that v_i is adjacent to w_j, w_{j+1} and w_{j+2}, contradicting item (2) of Lemma 10, hence $l > 2$.

Since $w_j, w_{j+1}, w_{j+3} \in I$ and $v_{i-1}, v_i, v_{i+1} \notin I$, we have that $w_{j+2} \neq v_{i-2}$ and $w_{j+2} \neq v_{i+1}$. Also, since $w_{j+1}, v_{i-2}, v_{i+1}$ induce a triangle, Lemma 11 implies that $w_{j+2}, v_{i-2}, v_{i+1}$ induce a triangle or w_{j+2} is adjacent to neither v_{i-2} nor v_{i+1}.

If $w_{j+2}, v_{i-2}, v_{i+1}$ induce a triangle, since v_{i+1} is adjacent to both w_{j+1} and w_{j+2}, by item 2 of Lemma 10 it follows that v_{i+1} is non-adjacent to w_{j+3}. In this case, we have $l > 3$. By the same arguments as before (interchanging $j+2$ and $j+3$) we conclude that $w_{j+3} \neq v_{i-2}$ and $w_{j+3} \neq v_{i+1}$. By Lemma 11, knowing that w_{j+3} is non-adjacent to v_{i+1}, it follows that w_{j+3} is adjacent to neither v_{i-2} nor v_{i+1}. So, we conclude that if $w_{j+2}, v_{i-2}, v_{i+1}$ induce a triangle then w_{j+3} is adjacent to neither v_{i-2} nor v_{i+1}.

If $w_{j+2}, v_{i-2}, v_{i+1}$ induce a triangle, define $a = 3$ and, if w_{j+2} is not adjacent to none of v_{i-2}, v_{i+1}, define $a = 2$. In both cases ($a = 2$ or $a = 3$), w_{j+a} is adjacent to neither v_{i-2} nor v_{i+1}; $w_{j+a-1}, v_{i-2}, v_{i+1}$ induce a triangle, and $a < l$. Then, by Lemma 14, w_{j+a}, v_{i-1}, v_i induce a triangle. This is a contradiction, since the triangles w_{j+a}, v_{i-1}, v_i and w_{j+l}, v_i, v_{i+1} contradict the minimality of $l = |h|$ on the election of j and h (taking into account that the distance between w_{j+a} and w_{j+l} is $l - a$). □

Lemma 17 *Let G be a perfect $\{$gem,W_4,bull$\}$-free graph and $C = v_1, \ldots, v_{2k+1}$ ($k \geq 2$) be an intersection cycle of a hole of cliques of G. Then C does not contain two consecutive proper edges.*

PROOF. Suppose the lemma is false. Then, there exist vertices v_{i-1}, v_i, v_{i+1} such that (v_{i-1}, v_i) and (v_i, v_{i+1}) are edges which do not belong to any triangle containing only vertices of C. By Lemma 15, v_i is not lonely and therefore there are vertices v_{i-j}, v_{i+l} such that v_{i-j}, v_i, v_{i+l} is a triangle. We may assume that we have chosen $l \geq 1$ to be minimum and then (once l is chosen) we choose $j \geq 1$ to be minimum. We may also assume, changing the labels of the vertices of C if necessary, that $j \geq l$ and $i - j < i < i + l$. Therefore, the sets $\{i - j, i - j + 1, \ldots, i - 1\}$ and $\{i + 1, i + 2, \ldots, i + l\}$ do not intersect.

Since (v_i, v_{i+1}) is proper, it follows that neither v_{i-j}, v_i, v_{i+1} nor v_i, v_{i+1}, v_{i+l} is a triangle, so v_{i+1} is adjacent to none of v_{i-j}, v_{i+l}. Therefore, $l > 1$. Neither v_{i+l-1}, v_i, v_{i-j} nor v_{i+l}, v_i, v_{i-j+1} are triangles because we have chosen l minimum and then we have taken j minimum. Therefore, by Lemma 11 (setting $i := i + l$, $l := i$, $j := i - j$ and $d := -1$) v_{i+l-1} is adjacent to neither v_i nor v_{i-j} and (setting $i := i - j$, $l := i + l$, $j := i$ and $d := 1$) v_{i-j+1} is adjacent to neither v_i nor v_{i+l}. Since v_{i+1} is adjacent to neither v_{i+l} nor v_{i-j}, Lemma 12 implies that $v_{i+1}, v_{i+l-1}, v_{i-j+1}$ is a triangle. Labelling the vertices of C in the reverse order and interchanging j and l it follows that $v_{i-1}, v_{i+l-1}, v_{i-j+1}$ is also a triangle (note that the conditions for l and j are not symmetric, but in the argument above we have used them in a symmetric way).

By item (1) of Lemma 10, C has no short chord, so $l > 2$. Now we split our proof into two cases, either: 1) $l = j = 3$ or 2) $j > 3$, $l \geq 3$.

Case 1) $l = j = 3$: In this case $v_{i+1}, v_{i+2}, v_{i-2}$ is a triangle and $v_{i-1}, v_{i+2}, v_{i-2}$ is another triangle. Since $Q = Q(v_{i-2}, v_{i-1})$ is a clique and v_{i-2}, v_{i-1} are both adjacent to v_{i+2}, there exists a vertex $w \in Q - \{v_{i-1}, v_{i-2}\}$ non-adjacent to v_{i+2}. The cycle C has no short chord, so v_{i-1} is non-adjacent to v_{i+1}. Therefore, $w, v_{i-1}, v_{i+2}, v_{i+1}$ induce a hole or a path. Furthermore, v_{i-2} is adjacent to all of them, so these five vertices induce a gem or W_4, which is a contradiction.

Case 2) $l \geq 3$, $j > 3$: By Lemma 11 (instantiating $i := i - j + 1$, $j = i + 1$, $l = i + l - 1$ and $d := 1$), v_{i-j+2} is adjacent to both v_{i+1} and v_{i+l-1} (case 2A) or to none of them (case 2B). In case 2A, by item (2) of Lemma 10, as v_{i+l-1} is adjacent to both v_{i-j+1} and v_{i-j+2}, v_{i+l-1} is non-adjacent to v_{i-j+3}. Similarly, we obtain that v_{i-j+3} is non-adjacent to v_{i+1}.

Let $a = j - 3$ in case 2A, and $a = j - 2$ in case 2B. In both cases v_{i-a-1}, v_{i+l-1}, v_{i+1} is a triangle and v_{i-a} is not adjacent to neither v_{i+l-1} nor v_{i+1}. If v_{i+l} is adjacent to v_{i-a-1}, since v_{i+l-1} is also adjacent to v_{i-a-1} and $Q' = Q_C(v_{i+l}, v_{i+l-1})$ is a clique, it follows that there is a vertex $w \in Q'$ which is non-adjacent to v_{i-a-1}. Recalling that v_{i+l} is non-adjacent to v_{i+1}, we obtain that $v_{i+l-1}, w, v_{i+l}, v_{i-a-1}, v_{i+1}$ induce a gem or W_4, which is a contradiction. So, v_{i+l} is non-adjacent to v_{i-a-1}.

We already know that $v_{i-a-1}, v_{i+l-1}, v_{i+1}$ is a triangle and v_{i-a} is adjacent to neither v_{i+l-1} nor v_{i+1}; v_{i+l} is adjacent to neither v_{i-a-1} nor v_{i+1}; and, as (v_i, v_{i+1}) is proper, v_i is adjacent to neither v_{i+l-1} nor v_{i-a-1}. By Lemma 12, v_{i-a}, v_{i+l}, v_i is a triangle, which is a contradiction because $a < j$ and we have taken j to be minimum. □

We can now prove the main results of this section.

Theorem 18 *If G is a perfect $\{$gem,W_4,bull$\}$-free graph then G is K-perfect.*

PROOF. Suppose G is not K-perfect. By Corollary 9, $K(G)$ contains no odd antihole of length greater than 5. Therefore, $K(G)$ contains an odd hole, and in consequence there exists an odd hole of cliques in G. So there is an odd-length intersection cycle $v_1, \ldots, v_{2k+1}$ in G ($k \geq 2$). Call $e_i = (v_i, v_{i+1})$ for all $1 \leq i \leq 2k + 1$. By Lemmas 16 and 17 we may assume that e_1 is an improper edge and e_2 is a proper edge. By a repeated application of Lemmas 16 and 17 (note that the cycle is odd) we obtain that e_{2k+1} is improper and therefore e_1 is proper, a contradiction. □

Theorem 19 *Let G be a $\{$gem,W_4,bull$\}$-free graph. Then the following statements are equivalent:*

(i) *G is perfect.*
(ii) *G is clique-perfect.*
(iii) *G is coordinated.*

PROOF. This is a direct corollary of Theorem 18 and the fact that every graph in a hereditary class of K-perfect clique-Helly graphs, is clique-perfect and coordinated. Recall that $\{$gem,$W_4\}$-free graphs are a hereditary class of clique-Helly graphs and the only clique-perfect graphs which are minimally imperfect ($\overline{C_{6j+3}}$, for $j \geq 1$) contain gems. □

Corollary 20 *The clique-perfect and coordinated graph recognition problem restricted to the class of* $\{gem, W_4, bull\}$*-free graphs can be solved in polynomial time.*

PROOF. It is a direct consequence of Theorem 19 and the fact that perfect graphs can be recognized in polynomial time [6]. □

3 Summary

These results allow us to formulate partial characterizations of clique-perfect and coordinated graphs by minimal forbidden subgraphs on two superclasses of triangle-free graphs, as it is shown in Table 1.

Graph classes	**Forbidden subgraphs**	**Recognition**	**Ref.**
Paw-free graphs	odd holes	linear	Thm 6
{gem,W_4,bull}-free graphs	odd holes	polynomial	Thm 19

Table 1
Minimal forbidden induced subgraphs for clique-perfect and coordinated graphs in each class analyzed here.

It remains as an open problem to determine the "biggest" superclass of triangle-free graphs where the three classes studied here (perfect, clique-perfect and coordinated graphs) are equivalent.

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