International Journal of Computer Mathematics
Vol. 00, No. 00, January 2009, 1–8

RESEARCH ARTICLE

On the $L(2,1)$-labeling of block graphs*

Flavia Bonomo†a Márcia R. Cerioli‡b *

a *CONICET and Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina. e-mail: fbonomo@dc.uba.ar*
b *Instituto de Matemática and COPPE/Sistemas e Computação, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil. e-mail: cerioli@cos.ufrj.br*

(*Received 00 Month 200x; in final form 00 Month 200x*)

The distance-two labeling problem of graphs was proposed by Griggs and Roberts in 1988, and it is a variation of the frequency assignment problem introduced by Hale in 1980. An $L(2,1)$-labeling of a graph G is an assignment of nonnegative integers to the vertices of G such that vertices at distance two receive different numbers and adjacent vertices receive different and nonconsecutive integers. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest integer k such that G has a $L(2,1)$-labeling in which no label is greater than k.

In this work we study the $L(2,1)$-labeling problem on block graphs. We find upper bounds for $\lambda(G)$ in the general case, and we reduce those bounds for some particular cases of block graphs with maximum clique size equal to 3.

Keywords: block graphs, distance-two labeling problem, graph coloring.

AMS Subject Classification: 05C15
ACM Computing Classification System: G.2.2 - Graph labeling.

1. Introduction

The distance-two labeling problem of graphs was proposed by Griggs and Roberts in 1988 (c.f. [7]), and it is a variation of the frequency assignment problem introduced by Hale in 1980 [8]. Suppose we are given a number of transmitters or stations. The $L(2,1)$-labeling problem addresses the problem of assigning frequencies (nonnegative integers) to the transmitters so that "close" transmitters receive different frequencies and "very close" transmitters receive frequencies that are at least two frequencies apart.

Let G be a simple, finite, undirected graph with vertex set $V(G)$. Let $\Delta(G)$ denote the maximum degree of a vertex of G, $d_G(u,v)$ denote the distance in G between vertices u and v, and $\omega(G)$ denote the maximum size of a clique of G.

Let k be a nonnegative integer. Denote by $[0,k]$ the set $\{x \in \mathbb{Z} : 0 \leq x \leq k\}$.

An $L(2,1)$-*labeling* of a graph G is a function $f : V(G) \to \mathbb{Z}_{\geq 0}$ such that $|f(u) - f(v)| \geq 1$ if $d_G(u,v) = 2$ and $|f(u) - f(v)| \geq 2$ if $d_G(u,v) = 1$. For a nonnegative

*This work was partially supported by the CAPES/SPU Project CAPG-BA 008/02.
†Partially supported by ANPCyT PICT-2007-00518 and PICT-2007-00533, and UBACyT Grants X069 and X606 (Argentina).
‡Partially supported by CNPq and FAPERJ (Brazil).

*Corresponding author. Email: fbonomo@dc.uba.ar

ISSN: 0020-7160 print/ISSN 1029-0265 online

DOI: 0020716YYxxxxxxxx
http://www.informaworld.com

integer k, a k-$L(2,1)$-*labeling* is an $L(2,1)$-labeling $f : V(G) \to [0,k]$. The $L(2,1)$-*labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labeling. It is not difficult to see that $\lambda(G) \geq \Delta(G)+1$, and $\lambda(G) \geq 2\omega(G)-2$. The $L(2,1)$-labeling problem has been studied widely. Griggs and Yeh [7] showed that the $L(2,1)$-labeling problem is NP-complete for general graphs. They proved that $\lambda(G) \leq \Delta^2(G)+2\Delta(G)$ and conjectured that $\lambda(G) \leq \Delta^2(G)$ for general graphs different from K_2. Chang and Kuo [1] proved that $\lambda(G) \leq \Delta^2(G)+\Delta(G)$ and gave a linear-time algorithm for the $L(2,1)$-labeling problem on cographs. Král and Škrekovski [10] proved that $\lambda(G) \leq \Delta^2(G)+\Delta(G)-1$ for graphs different from K_2. More recently, Gonçalves [6] proved that $\lambda(G) \leq \Delta^2(G)+\Delta(G)-2$, giving the best known upper bound for small values of $\Delta(G)$. In [9], the authors prove Griggs and Yeh's conjecture for graphs G with $\Delta(G)$ sufficiently large. For further studies on the $L(2,1)$-labeling and some generalizations, see [2–6, 11, 12].

A *block* of a graph is a maximal 2-connected component. An *end-block* is a block containing exactly one cutpoint. A *block-cutpoint tree* of a graph G is a tree whose vertices are the cutpoints and the blocks of G, such that two vertices are adjacent if and only if they correspond to a block B of G and a cutpoint v of B. A graph is a *block graph* if it is connected and every block is a clique.

Block graphs with $\omega(G)=2$ are trees. Griggs and Yeh [7] showed that $\Delta(G)+1 \leq \lambda(G) \leq \Delta(G)+2$ for trees, and Chang and Kuo [1] gave a polynomial-time algorithm for the $L(2,1)$-labeling problem on this class of graphs. However, there is no simple characterization distinguishing the cases $\lambda = \Delta+1$ and $\lambda = \Delta+2$. For the special case of paths, it is not difficult to see that $\lambda(P_1)=0$, $\lambda(P_2)=2$, $\lambda(P_3)=\lambda(P_4)=3$ and $\lambda(P_n)=4$ for $n \geq 5$.

The aim of this work is to study the $L(2,1)$-labeling problem on block graphs. We find upper bounds for $\lambda(G)$ in the general case, and we reduce those bounds for some particular cases with $\omega(G)=3$.

2. Upper bounds

THEOREM 2.1 *Let G be a block graph with maximum degree Δ and maximum clique size ω. Then $\lambda(G) \leq \max\{\Delta+2, \min\{3\omega-2, \Delta+\omega\}\}$.*

Proof Let G be a block graph with maximum degree Δ and maximum clique size ω, and let $k = \max\{\Delta+2, \min\{3\omega-2, \Delta+\omega\}\}$. We will prove that G has a k-$L(2,1)$-labeling by induction on the number of blocks. If G is a complete graph of n vertices, then $\omega = n$, $\Delta = n-1$ and $k = \max\{n+1, \min\{3n-2, 2n-1\}\} = \max\{n+1, 2n-1\} \geq 2n-2 = \lambda(G)$. Suppose now that G is not a complete graph, and let v be a cutpoint of G such that all the blocks containing v but at most one are end-blocks. Denote by $B_1, B_2, \ldots, B_t$ the blocks containing v, where $B_2, \ldots, B_t$ are end-blocks with $|B_2| \geq \cdots \geq |B_t|$, let $B'_i = B_i - v$ for $i = 1, \ldots, t$ and let $B = \bigcup_{2 \leq i \leq t} B'_i$. Let $b_i = |B'_i|$ for $i = 1, \ldots, t$ and let $b = |B|$. By inductive hypothesis, there is a $L(2,1)$-labeling of $G \setminus B$ with labels in $[0,k]$. We will extend this labeling to B. From the set $[0,k]$, B can use neither the label used by v nor its previous or subsequent label, and it cannot use any of the labels used by the neighbors of v in $G \setminus B$, that is, vertices in B'_1. So the available labels for B are at least $k+1-3-b_1$. One can observe that the labeling can be extended to B if there are at least $\max\{b, 2b_2-1\}$ available labels. In fact, let $c_1 < c_2 < \cdots < c_p$ be the available labels. The vertices in B_2, and subsequently in $B_3, \ldots, B_t$ can be labeled by using first labels with odd indices followed by the even ones, respecting the increasing ordering of labels. Since $p \geq 2b_2-1$, vertices in the same set B_i do not receive consecutive labels. It holds that $k-2-b_1 \geq \max\{b, 2b_2-1\}$ if and only if $k \geq \max\{b+b_1+2, b_1+2b_2+1\}$. Note that $b_1+b \leq \Delta$, and for $i=1,\ldots,t$, $b_i \leq \omega - 1$ since $B'_i \cup \{v\}$ is a clique. Thus $b_1+b+2 \leq \Delta+2$, $b_1+2b_2+1 \leq 3(\omega-1)+1 = 3\omega-2$, and $b_1+2b_2+1 \leq (b_1+b_2)+b_2+1 \leq \Delta+b_2+1 \leq \Delta+\omega$. Therefore, $k = \max\{\Delta+2, \min\{3\omega-2, \Delta+\omega\}\} \geq \max\{b+b_1+2, b_1+2b_2+1\}$. This completes the proof. ■

Corollary 2.2 *Let G be a block graph different from K_2. Then $\lambda(G) \leq 2\Delta(G)+1$ and $\lambda(G) \leq \Delta^2(G)$.*

Proof If G is a block graph different from K_2, then $\Delta(G) \geq 2$. So, $\Delta(G)+2 \leq \Delta^2(G)$. If $\Delta(G)=2$ then G is either a path or a triangle, and in both cases it is known that $\lambda(G) \leq 4$. If $\Delta(G) \geq 3$, then $\Delta(G)+\omega(G) \leq 2\Delta(G)+1 \leq \Delta^2(G)$. Thus $\lambda(G) \leq \max\{\Delta(G)+2, \min\{3\omega(G)-2, \Delta(G)+\omega(G)\}\} \leq \min\{2\Delta(G)+1, \Delta^2(G)\}$. ■

Corollary 2.3 *Let G be a block graph with maximum degree Δ and maximum clique size at most 3. If $\Delta \geq 5$ then $\lambda(G) \leq \Delta+2$, if $\Delta = 4$ then $\lambda(G) \leq 7$, and if $\Delta \leq 3$ then $\lambda(G) \leq 6$.*

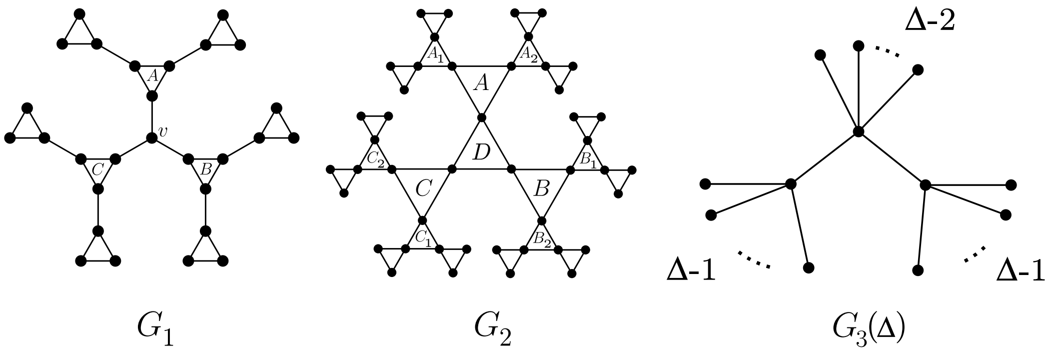

Figure 1. Examples showing tightness of the bounds of Corollary 2.3.

Proposition 2.4 *The bounds of Corollary 2.3 are tight for $\Delta = 3$, $\Delta = 4$ and $\Delta \geq 5$, and they are attained by graphs G_1, G_2 and $G_3(\Delta)$ of Figure 1, respectively.*

Proof Let us consider 5-$L(2,1)$-labelings, and show that G_1 does not admit one. In a 5-$L(2,1)$-labeling of a graph, the set of possible triplets for a triangle is $\{$ 0, 2, 4; 0, 2, 5; 0, 3, 5; 1, 3, 5 $\}$. Let $A_1 = v_1v_2v_3$ and $A_2 = v_4v_5v_6$ be two disjoint triangles in a graph, joined by the edge v_1v_4. If A_1 is labeled 0, 2, 5 then v_1 cannot receive number 2, otherwise v_4 must receive number 4 but A_2 cannot be labeled 0, 2, 4, and v_1 cannot receive number 5, otherwise v_4 must receive number 3 or 1, but A_2 can neither be labeled 0, 3, 5 nor 1, 3, 5. Analogously, if A_1 is labeled 0, 3, 5, then v_1 cannot receive numbers 0 or 3. Therefore, in a 5-$L(2,1)$-labeling of G_1 the triangles A, B and C should be labeled 0, 2, 4 or 1, 3, 5. If two of A, B, C use different labels then there is no color left to v. If all of them use the same labeling, the three neighbors of v must use different colors. Since none of 1, 3, 5 is at distance two of 0, 2 and 4, and conversely, there is no suitable color for v. Consider the graph G_2 in Figure 1, and suppose there is an $L(2,1)$-labeling of it with labels in $[0,6]$. The set of possible triplets for labeling a triangle is $\{$ 0, 2, 4; 0, 2, 5; 0, 2, 6; 0, 3, 5; 0, 3, 6; 0, 4, 6; 1, 3, 5; 1, 3, 6; 1, 4, 6; 2, 4, 6 $\}$. We say that two triplets are *compatible* if they share exactly one number, and we say that a triplet is *good* if it has three compatible triplets, each one sharing a different number with it. As it can be seen in Figure 2, the set of good triplets is $\{$ 0, 2, 5; 0, 3, 6; 1, 3, 5; 1, 4, 6 $\}$. It is clear that the triangles A, B, C, D, A_1, A_2, B_1, B_2, C_1 and C_2 of G_2 should be labeled with good triplets. So, we call *very good* triplets those triplets having three compatible good triplets, each one sharing a different number. As it can be seen in Figure 2, the only very good triplets are $0, 3, 6$ and $1, 3, 5$. The triangles A, B, C and D of G_2 should be labeled with very good triplets. Since the vertices of D are labeled by very good triplets and $\{0, 3, 6\} \cap \{1, 3, 5\} = \{3\}$, at least one of A, B or C are labeled by the same very good triplet as for D, a contradiction. Hence, the $L(2, 1)$-labeling number of G_2 is greater than 6. The family of graphs $G_3(\Delta)$ is a known example of trees with maximum degree Δ and $L(2, 1)$-labeling number $\Delta + 2$: in every $L(2, 1)$-labeling of a graph with labels in $[0, \Delta + 1]$, all the vertices of degree Δ must receive label 0 or $\Delta + 1$, but the three vertices of degree Δ in $G_3(\Delta)$ must receive pairwise distinct labels. ■

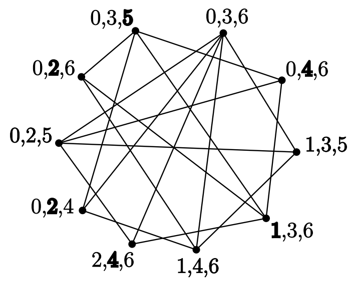

Figure 2. Compatibility between triplets on $[0, 6]$.

Let T_3 be the leftmost graph in Figure 3, that is, a graph with seven vertices $v_1, v_2, v_3, v_4, w_1, w_2, w_3$ where v_1, v_2, v_3, v_4 induce a path of length four, and w_i is adjacent to v_i and v_{i+1} for $i = 1, 2, 3$.

THEOREM 2.5 *Let G be a block graph with maximum degree* 4 *and maximum clique size* 3*. If G does not contain T_3, then $\lambda(G) \leq 6$.*

Proof We will construct a 6-$L(2, 1)$-labeling of G by labeling the vertices of G ordered by their distance to some vertex v_0 with degree 4. For $d \geq 0$, denote by V_d the set of vertices of G at distance d from v_0. Please note that since G is a block graph with $\omega(G) = 3$, each vertex in V_d with $d \geq 1$ has exactly one neighbor in V_{d-1}, and at most one neighbor in V_d and, in that case, they share the neighbor in V_{d-1}. We call *type 1* the vertices belonging to two triangles. Note that a type 1 vertex belongs to a triangle formed by it and its neighbors in V_d and V_{d-1}, and to another triangle formed by it and its neighbors in V_{d+1}. We call *type 2* those vertices that are not of type 1. As in the proof of Proposition 2.4, we will consider the set of possible triplets for labeling a triangle. Recall that two triplets are *compatible* if they share exactly one number, and we say that a number a is *bad* for a triplet t if t has no compatible triplet sharing the number a. As it can be seen in Figure 2, every triplet has at most one bad number for it, which is boldfaced in each triplet.

First, we give to v_0 the label 0. Now, we will continue the labeling process in such a way that no type 1 vertex in V_d, with $d \geq 1$, is labeled with a bad color for the triplet given to the triangle formed by it and its neighbors in V_d and V_{d-1}.

At most one of the four vertices in V_1 is of type 1, otherwise G would contain T_3 as an induced subgraph, so we can label vertices in V_1 following that rule. Let $d > 1$ and suppose that every vertex at distance at most $d - 1$ from v_0 is labeled. Let v be a no-labeled vertex in V_d, and w its neighbor in V_{d-1}. Let W be the set of neighbors of w in V_d. We will label the vertices in W, all at once. Since w has a neighbor in V_{d-2} and G has maximum degree 4, $|W| \leq 3$. If w has no neighbor in V_{d-1}, then the number of available colors for W is at least 3. If the vertices in W are pairwise non-adjacent, we can clearly label them. If two of them are adjacent, at most one is type 1, otherwise G would contain T_3 as an induced subgraph. So we can label those two vertices with two non-consecutive numbers within the available ones, in such a way that the (possible) type 1 vertex does not receive a bad color for the triplet formed by these two labels and the label of w. Finally, there is at least one remaining label for the (possible) third vertex. If w has a neighbor in V_{d-1}, then $|W| \leq 2$ and the number of available colors for W is at least 2. If the vertices in W are non-adjacent, we can clearly label them. If they are adjacent, then w is type 1 and none of the vertices in W is, otherwise G would contain T_3 as an induced subgraph. Since w is not labeled with a bad color for the triplet given to the triangle formed by it and its neighbors in V_{d-1} and V_{d-2}, there is a triplet compatible to that one, in order to label W. ■

By now, the computational complexity of computing $\lambda(G)$ of a block graph G is open, even when $\omega(G) = 3$. Nevertheless, the proofs of the previous theorems are constructive, and lead to algorithms to produce an $L(2,1)$-labeling of the graph with the showed upper bound.

3. The $L(2,1)$-labeling number for paths of triangles

We call *paths of triangles* the block graphs G with $\omega(G) = 3$ and such that the block-cutpoint tree of G is a path. Examples of paths of triangles can be seen in Figure 3. Please note that, since $\omega(G) = 3$, then $\lambda(G) \geq 4$.

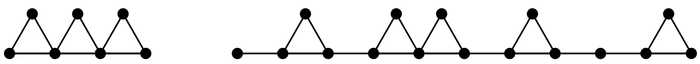

Figure 3. Paths of triangles.

For these kind of graphs we prove that $\lambda(G) \leq \Delta(G) + 2$ and give a complete characterization for each possible value of λ.

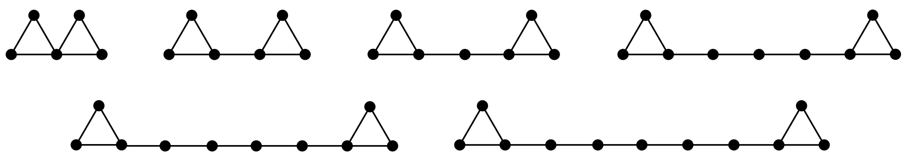

Figure 4. Paths of triangles with $\lambda = 5$.

THEOREM 3.1 *Let G be a path of triangles. Then $\lambda(G) \leq 6$. Moreover, $\lambda(G) = 6$ if and only if G contains T_3, and $\lambda(G) = 4$ if and only if G does not contain any of the graphs in Figure 4.*

Proof The scheme in Figure 5 shows that for every path of triangles G, $\lambda(G) \leq 6$.

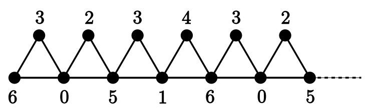

Figure 5. Scheme for the 6-$L(2,1)$-labeling on paths of triangles.

It is easy to see also that T_3 does not admit a 5-$L(2,1)$-labeling, because the only two pairs of compatible triplets are $0,2,5$ and $1,3,5$ sharing 5, and $0,2,4$ and $0,3,5$ sharing 0.

We will show that every path of triangles with no induced T_3 can be 5-$L(2,1)$-labeled. In order to do that, we will consider a path of triangles without T_3 as a sequence of pieces consisting on one or two consecutive triangles joined by simple paths. We will consider four different ways of labeling a piece consisting on two consecutive triangles, namely A, B, C and D, and four different ways of labeling a piece consisting on a sigle triangle, namely A', B', C' and D'. Figure 6 shows A, B, C, D, A', B', C' and D', together with the possible labels for the last vertex in a path preceding the piece and the first vertex in a path succeeding the piece. In the scheme on the right of Figure 6, we have a node for every way of labeling a piece, and we join a node X and a node Y with a directed arc labeled t to mean that a piece labeled as X can be succeeded by a piece labeled as Y with a join path of length t. We omit the arcs between A, B, C, D and A', B', C', D', but note that the first two vertices of X and X' are identically labeled. Thus, if we have an arc (X, Y) labeled by t then we could add an arc (X, Y') labeled by t, and if we have an arc (X', Y') labeled by t then we could add an arc (X', Y) labeled by t. Since for every length t and for every node X there exists a directed arc labeled t joining X with one of A, B, C, D (and, consequently, with one of A', B', C', D'), every path of triangles without T_3 can be 5-$L(2,1)$-labeled, subject to the correctness of the scheme. The arcs labeled up to 5 are easy to check by hand. It remains to prove that B and A (resp. D and C) can be joined to A (resp. C) by a path of arbitrary length at least 5, and that B' and A' (resp. D' and C') can be joined to A' (resp. C') by a path of arbitrary length at least 2.

To simplify the notation, we will enclose with brackets a subsequence of a sequence to mean that it can be either omitted or repeated as many times as necessary. For example, the sequence $1, 2, [3, 4], 5$ will stand for any of the sequences $1, 2, 5$, $1, 2, 3, 4, 5$, $1, 2, 3, 4, 3, 4, 5$, etc..

<u>Case 1:</u> We have to join $5, 1$ or $5, 2$ with $5, 2$ by paths of length at least 5. We will use subsequences of the 5-periodic sequence $[2, 4, 1, 3, 0]$ in the following way: if the length of the path is $5k$, $5k + 1$ or $5k + 2$ we will join $5, 2$ with $5, 2$ by using the subsequence beginning at 4 (for example, $5, 2, 4, 1, 5, 2$; $5, 2, 4, 1, 3, 5, 2$; $5, 2, 4, 1, 3, 0, 5, 2$; $5, 2, 4, 1, 3, 0, 2, 4, 1, 5, 2$; etc.); if the length of the path is $5k + 3$ or $5k + 4$ we will join $5, 1$ with $5, 2$ by using the sequence beginning at 3 (for example, $5, 1, 3, 0, 2, 4, 1, 5, 2$; $5, 1, 3, 0, 2, 4, 1, 3, 5, 2$; $5, 1, 3, 0, 2, 4, 1, 3, 0, 2, 4, 1, 5, 2$; etc.).

<u>Case 2:</u> We have to join $0, 4$ or $0, 3$ with $0, 3$ by paths of length at least 5. This case is symmetric of Case 1, considering the isometric bijection between labels $t \mapsto 5 - t$.

<u>Case 3:</u> We have to join $0, 3$ or $0, 5$ with $5, 2$ by paths of length at least 2. The firsts cases are $0, 5, 2$; $0, 3, 5, 2$; $0, 3, 1, 5, 2$. For paths of length greater than 5 we will join $0, 5$ with $5, 2$ by using subsequences of the 5-periodic sequence $[2, 4, 1, 3, 0]$ in the following way: if the length of the path is $5k$ or $5k+4$ we will use the sequence beginning at 1 (for example, $0, 5, 1, 3, 5, 2$; $0, 5, 1, 3, 0, 2, 4, 1, 5, 2$; $0, 5, 1, 3, 0, 2, 4, 1, 3, 5, 2$; etc.) and if the length of the path is $5k+1$, $5k+2$ or $5k+3$ we will use the sequence beginning at 2 (for example, $0, 5, 2, 4, 1, 5, 2$; $0, 5, 2, 4, 1, 3, 5, 2$; $0, 5, 2, 4, 1, 3, 0, 5, 2$; etc.).

<u>Case 4:</u> We have to join $5, 2$ or $5, 0$ with $0, 3$ by paths of length at least 2. This case is symmetric of Case 3, considering the isometric bijection between labels $t \mapsto 5 - t$.

Finally, we will characterize the paths of triangles G with $\lambda(G) = 4$. Since $\lambda(G) \geq \Delta(G)+1$, then $\Delta(G) \leq 3$. In particular, G cannot contain the first graph in Figure 4. Moreover, if $\Delta(G) = 3$ then every vertex of degree three should be labeled with 0 or 4. The only possible triplet in this case is $0, 2, 4$, and by the observation above, in every triangle the label 2 should be assigned to a vertex of degree two. Considering these facts, it is not difficult to check that none of the graphs in Figure 4 admits

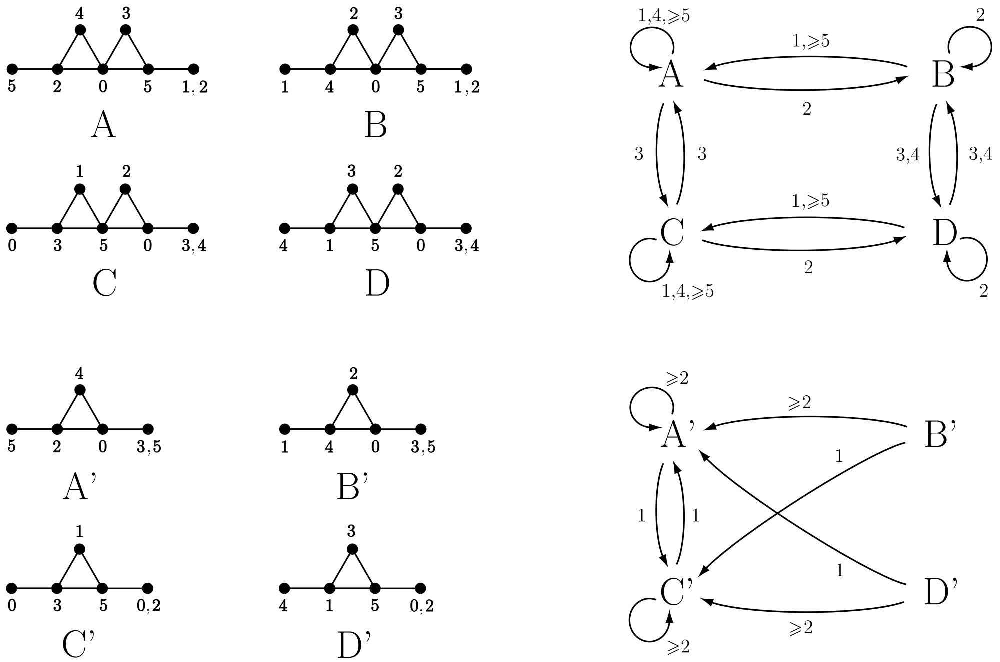

Figure 6. Scheme for the 5-$L(2,1)$-labeling of paths of triangles without T_3.

a 4-$L(2,1)$-labeling.

Since 4-$L(2,1)$-labelings are preserved under the map $t \mapsto 4-t$, we can assume that the cutpoint in the first triangle linked to the path is labeled 0. Two triangles joined by a path of length 3 can be labeled giving to the vertices of degree three numbers 0 and 4, and to their neighbors in the path numbers 3 and 1, respectively. If we have two triangles joined by a path of length at least 7, we have to join $0,3$ with $3,0$ or $1,4$ by a path. If the length of the path is $3k+1$ we will join $0,3$ with $1,4$ by using the sequence $0,3,1,4,0,[2,4,0],3,1,4$. If the length of the path is $3k+2$ we will join $0,3$ with $1,4$ by using the sequence $0,3,1,4,2,0,[4,2,0],3,1,4$. Finally, if the length of the path is $3k$ we will join $0,3$ with $3,0$ by using the sequence $0,3,1,4,2,0,[4,2,0],4,1,3,0$. This completes the proof of the theorem. ∎

This characterization leads to an efficient algorithm to compute $\lambda(G)$ and an optimum $L(2,1)$-labeling on paths of triangles.

References

[1] G. Chang and D. Kuo, *The $L(2,1)$-labeling on graphs*, SIAM Journal on Discrete Mathematics 9 (1996), pp. 309–316.

[2] G. Chang and C. Lu, *Distance-two labeling of graphs*, European Journal of Combinatorics 24 (2003), pp. 53–58.

[3] G. Chang, W.T. Ke, D. Kuo, D.F. Lin, and R. Yeh, *On $L(d,1)$-labelings of graphs*, Discrete Mathematics 220 (2000), pp. 57–66.

[4] P. Fishburn and F. Roberts, *Full color theorems for $L(2,1)$-colorings*, SIAM Journal on Discrete Mathematics 20 (2006), pp. 428–443.

[5] J. Georges and D. Mauro, *On regular graphs optimally labeled with a condition at distance two*, SIAM Journal on Discrete Mathematics 17 (2003), pp. 320–331.

[6] D. Gonçalves, *On the $L(p,1)$-labelling of graphs*, Discrete Mathematics 308 (2008), pp. 1405–1414.

[7] J. Griggs and R. Yeh, *Labeling graphs with a condition at distance two*, SIAM Journal on Discrete Mathematics 5 (1992), pp. 586–595.

[8] W. Hale, *Frequency assignment: theory and applications*, in *Proceedings of the IEEE*, vol. 68, 1980, pp. 1497–1514.

[9] F. Havet, B. Reed, and J.S. Sereni, *$L(2,1)$-labelling of graphs*, in *Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms*, January, 2008, pp. 621–630.

[10] D. Král and R. Škrekovski, *A theorem about the channel assignment problem*, SIAM Journal on Discrete Mathematics 16 (2003), pp. 426–437.

[11] D. Kuo and J. Yan, *On $L(2,1)$-labelings of cartesian products of paths and cycles*, Discrete Mathematics 283 (2004), pp. 137–144.

[12] D.F. Liu and R. Yeh, *On distance-two labelings of graphs*, Ars Combinatoria 47 (1997), pp. 13–22.