Vertex intersection graphs of paths on a grid: characterization within block graphs

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Abstract We investigate graphs that can be represented as vertex intersections of horizontal and vertical paths in a grid, the so called B_0-VPG graphs. Recognizing this class is an NP-complete problem. Although, there exists a polynomial time algorithm for recognizing chordal B_0-VPG graphs. In this paper, we present a minimal forbidden induced subgraph characterization of B_0-VPG graphs restricted to block graphs. As a byproduct, the proof of the main theorem provides an alternative certifying recognition and representation algorithm for B_0-VPG graphs in the class of block graphs.

1 Introduction

A *VPG representation* of a graph G is a collection of paths of the two-dimensional grid where the paths represent the vertices of G in such a way that two vertices of G are adjacent in G if and only if the corresponding paths share at least one vertex of the grid. A graph which has a VPG representation is called a *VPG graph*. In this paper, we consider the subclass B_0-VPG.

A B_0-*VPG representation* of G is a VPG representation in which each path in the representation is either a horizontal path or a vertical path on the grid. A graph is a B_0-*VPG graph* if it has a B_0-VPG representation. Thus, a B_0-VPG graph is the intersection graph of zero bend paths on a grid.

The class of B_0-VPG graphs form a subclass of the well-known *segment graphs*, or intersection graphs of straight-line segments in the plane. Indeed, the class of *k-DIR graphs* has been defined as the class of segment graphs in which the straight-line segments lie in at most k directions [7]. Therefore, it is easy to see that B_0-VPG graphs are equivalent to 2-DIR graphs.

Representations by intersections of paths on grids arise naturally in the context of circuit layout problems and layout optimization [9] where a layout is modelled as paths (wires) on a grid. Often one seeks to minimize the number of times a wire is bent [3,8] in order to minimize the cost or difficulty of production. Other times layout may consist of several layers where the wires on

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each layer are not allowed to intersect. This is naturally modelled as the coloring problem on the corresponding intersection graph.

The recognition problem is NP-complete for both VPG and B_0-VPG graphs (see [1] for more details about this and related results). Since all interval graphs are B_0-VPG graphs, it is natural to consider other subclasses of chordal B_0-VPG graphs. In [5], certain subclasses of B_0-VPG graphs have been characterized and shown to admit a polynomial time recognition; namely split, chordal claw-free and chordal bull-free B_0-VPG graphs. Recently, in [4] the authors present a polynomial time algorithm for deciding whether a given chordal graph is a B_0-VPG graph. In [1], it was shown that chordal B_0-VPG graphs are equivalent to the strongly chordal B_0-VPG graphs.

In this paper, we consider B_0-VPG graphs more from a structural point of view. We present a minimal forbidden induced subgraph characterization of B_0-VPG graphs restricted to block graphs. As a byproduct, the proof of the main theorem provides an alternative recognition and representation algorithm for B_0-VPG graphs in the class of block graphs.

2 Preliminaries

In this paper all graphs are connected, finite and simple. We use the notation that was used by Bondy and Murty [2].

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a graph G, let $|G|$ denote the cardinality of $V(G)$.

We write $G - v$ for the subgraph obtained by deleting a vertex v and all the edges incident to v. Similarly, for $A \subseteq V$, we denote by $G - A$ the subgraph of G obtained by deleting the vertices in A and all the edges incident to them, that is, $G - A = G[V \backslash A]$.

If H is a graph, a graph G is H-free if G contains no induced subgraph isomorphic to H. If $\mathcal{H}$ is a collection of graphs, the graph G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$.

A *clique* is a set of pairwise adjacent vertices which is maximal under inclusion. A *thin spider* N_n is the graph whose $2n$ vertices can be partitioned into a clique $K = \{c_1, ...c_n\}$ and a set $S = \{s_1, ..., s_n\}$ of pairwise nonadjacent vertices such that, for $1 \leq i, j \leq n$, s_i is adjacent to c_j if and only if $i = j$. We say that N_n is a thin spider of size n.

It was proved by Golumbic and Ries [5] that the thin spider N_5 is not B_0-VPG.

The following lemma will be used in our paper.

Lemma 1 *[1] In a B_0-VPG representation of a clique, all the corresponding paths share a common grid point.*

We will distinguish between two types of B_0-VPG representations of a clique: a *line clique* and a *cross clique*. We say that a clique is represented as a line clique if all paths corresponding to the vertices of the clique use a common row or a common column and intersect on at least one grid point of that row or column. A clique is said to be represented as a cross clique if the paths corresponding to the vertices of the clique share exactly one grid point, say (x_i, y_j), and there exists at least one such path which uses column x_i and at least one such path which uses row y_j. The grid point (x_i, y_j) is called the *center* of the cross clique (see Figure 1 for examples). It was observed in [5] that any B_0-VPG representation of a clique is either a line clique or a cross clique.
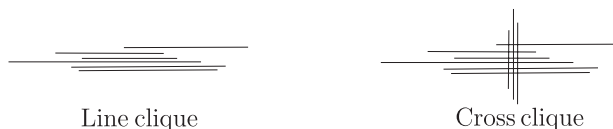

Line clique   Cross clique

Fig. 1 A line clique and a cross clique.

3 Block graphs

In this Section we will give a characterization of B_0-VPG graphs restricted to block graphs by a family of minimal forbidden induced subgraphs.

A *block graph* is a connected graph in which every two-connected component (*block*) is a clique. A *diamond* is a graph obtained from K_4 by deleting exactly one edge. A graph is called *chordal* if it does not contain any chordless cycle of length at least four. It is known that block graphs are connected chordal diamond-free graphs.

A *cutpoint* is a vertex whose removal from the graph increases the number of connected components.

Let G be a block graph. An *endblock* is a block having exactly one cutpoint. An *almost endblock* is a block B having at least two cutpoints and such that exactly one of these cutpoints belongs to blocks (different from B) that are not endblocks. An *internal block* is a block that is neither an endblock nor an almost endblock.

We denote by *3-cutpoint* a cutpoint that belongs to exactly 3 blocks, and by *2-cutpoint* a cutpoint that belongs to exactly 2 blocks, one of which is an endblock.

The *block-cutpoint-tree* $bc(G)$ of a graph G, introduced by Harary and Prins in [6], is a graph whose vertices are in one-to-one correspondence with the blocks and cutpoints of G, and such that two vertices of $bc(G)$ are adjacent if and only if one corresponds to a block H of G and the other to a cutpoint c of G, and c is in H.

Let $\mathcal{F}$ denote the family of block graphs obtained from N_5 by a finite sequence of applications of the following procedure: consider a complete subgraph of size 4 having at least two 2-cutpoints, say v_1 and v_2, with endblocks B_1 and B_2, respectively. Contract v_1 and v_2 into a single vertex x. Then, replace $B_1 - \{x\}$ and $B_2 - \{x\}$ by two thin spiders of size 3, making x adjacent to the vertices of the cliques of both the spiders. In Figure 2 we offer some examples of graphs in $\mathcal{F}$.
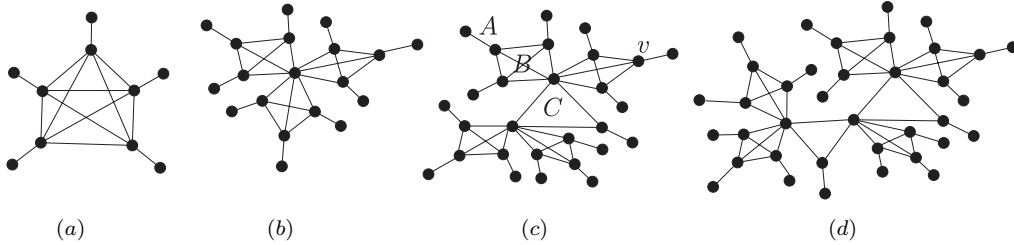

Fig. 2 Some examples of graphs in $\mathcal{F}$. In the figure, v is a cutpoint, A is an endblock, B is an almost endblock, and C is an internal block.

Proposition 2 *A graph in $\mathcal{F} - \{N_5\}$ has the following properties:*

  i. *each block is of size at most 4;*
 ii. *all the vertices are either leaves, 2-cutpoints or 3-cutpoints;*
iii. *the endblocks are of size 2 and have a 2-cutpoint;*
 iv. *the almost endblocks are of size 4 and have three 2-cutpoints and one 3-cutpoint;*
  v. *the internal blocks are of size 3 and have one 2-cutpoint and two 3-cutpoints;*
 vi. *a graph in $\mathcal{F}$ obtained from N_5 by applying the procedure k times, $k \geq 1$, has $6(k+1)$ blocks ($4(k+1)+1$ endblocks, $k+2$ almost endblocks, and $k-1$ internal blocks), $5(k+1)$ cutpoints (k 3-cutpoints and $4(k+1)+1$ 2-cutpoints), and $9(k+1)+1$ vertices.*

Proof We will prove it by induction on the number of times we apply the procedure. By symmetry of N_5, there is only one graph obtained by applying the procedure once (Figure 2(b)), and it has no internal blocks. It is easy to verify that this graph satisfies the properties claimed.

Suppose the properties are satisfied by all graphs in $\mathcal{F}$ obtained from N_5 by applying the procedure k times, $k \geq 1$, and let G be one such graph. Let us apply the procedure once more. Let

H be a complete subgraph of size 4 in G. By the inductive hypothesis, H is an almost endblock of G, and has three 2-cutpoints and one 3-cutpoint. By Proposition 2.iii, the blocks incident to the 3-cutpoint are not endblocks.

Choose two vertices v_1 and v_2 which are 2-cutpoints, and let B_1 and B_2 be the endblocks incident with v_1 and v_2, respectively. By Proposition 2.iii, B_1 and B_2 are of size 2. Contract v_1 and v_2 into a single vertex x, and replace $B_1 - \{x\}$ and $B_2 - \{x\}$ by two thin spiders of size 3, induced respectively by the vertices $\{c_1, c_2, c_3, s_1, s_2, s_3\}$ and $\{c_1', c_2', c_3', s_1', s_2', s_3'\}$, making x adjacent to the vertices of the cliques of both the spiders, i.e. $\{c_1, c_2, c_3, c_1', c_2', c_3'\}$.

After the procedure, $H' = H - \{v_1, v_2\} \cup \{x\}$ is a block of size 3, and it has two 3-cutpoints and still one 2-cutpoint. The new blocks $\{c_1, c_2, c_3, x\}$ and $\{c_1', c_2', c_3', x\}$ are almost endblocks, they are of size 4 and have three 2-cutpoints and one 3-cutpoint, namely x. And since the blocks incident to the other 3-cutpoints of H' are not endblocks, H' is an internal block. The six new endblocks $\{c_i, s_i\}$ and $\{c_i', s_i'\}$, $i = 1, 2, 3$ have a 2-cutpoint each (vertices c_i and c_i') and a leaf each (vertices s_i and s_i'). The remaining blocks, as well as their conditions, are not affected. So Proposition 2.(i–v) are satisfied by the new graph. To see Proposition 2.vi, notice that we have replaced 2 endblocks by 8 new blocks, 6 of which are endblocks and 2 of which are almost endblocks. Also, one almost endblock has become an internal block. We have replaced 4 vertices by 13 vertices and, in particular, two 2-cutpoints by one 3-cutpoint and six 2-cutpoints. □

Corollary 3 *The family $\mathcal{F}$ is countably infinite.*

Proof By Proposition 2, for every graph in $\mathcal{F}$ there is always an almost endblock on which we can perform the procedure in order to obtain a new graph in $\mathcal{F}$ with strictly more vertices. □

Corollary 4 *Each graph in $\mathcal{F}$ is minimal, i.e., it does not contain another graph in $\mathcal{F}$ as an induced subgraph.*

Proof Let $G \in \mathcal{F}$ and let G' be a proper connected induced subgraph of G. The blocks of G' are the blocks of G intersected with $V(G')$. Suppose $G' \in \mathcal{F}$, and suppose B' is a block of G' such that $B' = B \cap V(G')$, with B a block of G, and $|B'| < |B|$. Then, B cannot be an endblock of G because, by Proposition 2.iii, endblocks of G have size 2 and $|B'| < |B|$; B' cannot be an almost endblock of G' because by Proposition 2.i B has at most 4 vertices, and by Proposition 2.iv B' should have 4 vertices; B' cannot be an internal block of G' because, in that case, by Proposition 2 and the cardinalities of each type of block, B should be an almost endblock but, by Proposition 2.v, B' should have two 3-cutpoints while B has only one 3-cutpoint, and no 2-cutpoint of G may become a 3-cutpoint in an induced subgraph of it. So, B' is an endblock and B is either an almost endblock or an internal block. Let x be the cutpoint of B' in G'. By Proposition 2.iii, x is a 2-cutpoint of G'. If x is a 2-cutpoint in G, as B is not an endblock, we have that $G' = P_3$, and it does not belong to $\mathcal{F}$ (by Proposition 2.vi). If x is a 3-cutpoint in G, let B_1 and B_2 be the other two blocks in G that contain x. Since x is a 2-cutpoint in G', the intersection of one of these blocks with $V(G')$ is $\{x\}$. Without loss of generality, suppose this is the case of B_2. If B_1 is an almost endblock in G, then G' is an induced subgraph of the thin spider N_4, that is not in $\mathcal{F}$ (by Proposition 2.vi). If B_1 is an internal block, by cardinality, it may be either an endblock or an internal block in G'. In the first case, $G' = P_3$, that is not in $\mathcal{F}$. The second case cannot arise, because B_1 cannot have two 3-cutpoints in G' (no 2-cutpoint of G may become a 3-cutpoint in an induced subgraph of it). □

We will prove now some properties about the B_0-VPG representations of block graphs.

Lemma 5 *If a clique K of a block graph G has at least three cutpoints, then, in any B_0-VPG representation of G, it has to be represented as a cross clique.*

Proof Let v_i, $1 \le i \le 3$, be three of the cutpoints of K. Since v_i, $1 \le i \le 3$, are cutpoints there exist vertices x_j, $1 \le j \le 3$, such that v_i is adjacent to x_j if and only if $i = j$. Suppose that the clique K is represented as a line clique. Without loss of generality, we can assume that all the paths that represent vertices of K are horizontal paths using a common row of the grid. Suppose that P_{v_1} is the farthest line in the East direction (by *farthest line* in some direction, in the context

of a clique whose paths intersect at point p of the grid, we mean the path belonging to the clique and such that one of its endpoints maximizes the distance to p in that direction) and P_{v_2} is the farthest line using the West. But, P_{v_3} is a horizontal path lying in the same row that P_{v_1} and P_{v_2} and it has to be adjacent to P_{x_3}, and P_{x_3} is not adjacent with P_{v_1} and P_{v_3}. So, it is impossible to represent P_{x_3}. $\square$

Lemma 6 *If a clique K of a block graph G has four cutpoints, then, in any B_0-VPG representation of G, the four cutpoints are represented as the farthest lines South, North, West and East, respectively. Similarly, if a clique K has three cutpoints, then they are represented as the farthest lines of any three different cardinal points.*

Proof Suppose that K has four cutpoints. By Lemma 5, K has to be represented as a cross clique. Using the same idea that in the proof of Lemma 5, it is easy to see that the four cutpoints are represented as the farthest lines South, North, West and East, respectively.

In a similar way, it is easy to see that the result follows if K has three cutpoints. $\square$

Lemma 7 *In any B_0-VPG representation of the graph W, given in Figure 3, the intersection points of the cliques C_1, K, and C_2 lie in a same line of the grid, and the intersection point of the clique K lies between the intersection points of the cliques C_1 and C_2.*

Proof Let x_1, x_2, x_3 be the intersection points in the grid of the cliques C_1, K, C_2, respectively. Suppose, by the contrary, that there is a B_0-VPG representation of the graph W such that x_2 does not lie between x_1 and x_3. Without loss of generality, we can assume that x_2 is to the left of x_1 and x_3. By Lemmas 6 and 5, since C_1 and C_2 have four cutpoints, they are represented as cross cliques where the four cutpoints are the farthest lines South, North, West and East, respectively. So, it is impossible to represent the vertex w of W. $\square$

Remark 1 Observe that all the graphs in $\mathcal{F} - \{N_5\}$ have W as an induced subgraph.

Proof Let G be a graph in $\mathcal{F} - \{N_5\}$. Then G is obtained from a graph in $\mathcal{F}$ by the following procedure: consider a complete subgraph B_0 of size 4 having at least two 2-cutpoints, say v_1 and v_2, with endblocks B_1 and B_2, respectively. Contract v_1 and v_2 into a single vertex x. Then, replace $B_1 - \{x\}$ and $B_2 - \{x\}$ by two thin spiders of size 3, making x adjacent to the vertices of the cliques of both the spiders. The graph G has W as an induced subgraph, where the vertex labeled v in Figure 3 is the vertex x, the vertex labeled w in Figure 3 is a vertex of B_0 different from v_1 and v_2 (recall that B_0 has at least four vertices), and the remaining vertices in Figure 3 are the vertices of the two thin spiders of size 3. $\square$
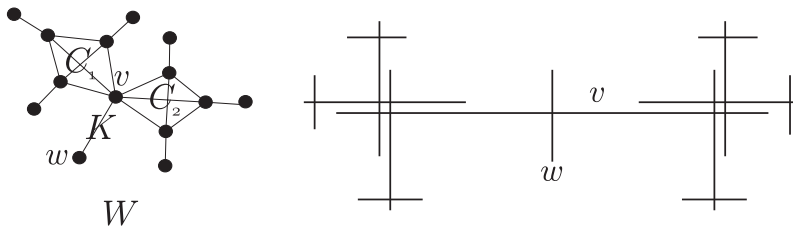

Fig. 3 The graph W and a B_0-VPG representation of it.

Lemma 8 *The graphs of $\mathcal{F}$ are not B_0-VPG.*

Proof The graph N_5 is not B_0-VPG [5]. We will proceed by induction on the number of applications of the procedure in the construction of the graph from N_5. Assume that if we applied the procedure k times, then we obtain a graph of $\mathcal{F}$ which is not B_0-VPG.

Let G be a graph of $\mathcal{F}$ which is obtained applying the procedure $k+1$ times. Suppose, on the contrary, that $G \in B_0$-VPG. We take a B_0-VPG representation of G.

By Remark 1, G has W as an induced subgraph. Let v be the vertex of W as in Figure 3, let P_v be the path which represents v in the B_0-VPG representation of G that we took. Let x_1, x_2, x_3 be the intersection points in the grid of the cliques C_1, K, C_2, respectively. Clearly, the three vertices lie in a same line of the grid and, by Lemma 7, x_2 lies between x_1 and x_3.

We are going to construct a new B_0-VPG representation. This is obtained from the previous one by removing the paths which correspond to C_1, C_2 and their corresponding endblocks; and adding the paths P_{v_i}, with $1 \leq i \leq 4$, such that $V(P_{v_1}) = \{x_1, x_2\}$, $V(P_{v_2}) = \{x_2, x_3\}$, $V(P_{v_3}) = \{x_1\}$ and $V(P_{v_4}) = \{x_3\}$. Observe that this is a B_0-VPG representation of a graph of $\mathcal{F}$ that was obtained applying the procedure k times, which is a contradiction.

Hence, the graphs of $\mathcal{F}$ are not B_0-VPG. □

We will prove the following theorem:

Theorem 9 *Let G be a block VPG graph. Then G is B_0-VPG if and only if G is $\mathcal{F}$-free. Moreover, the graphs of $\mathcal{F}$ are minimal not B_0-VPG.*

Proof The "only if" part follows from Lemma 8. For the "if" part, let G be a block $\mathcal{F}$-free graph. Let s be a BFS ordering of the vertices of the block-cutpoint-tree $bc(G)$, in such a way that s_1 is a vertex of $bc(G)$ corresponding to a block of G. Let H_i be the i-th block in s. We will consider the graph G_i as the graph induced by the first i blocks $H_1, \ldots, H_i$ in s, and proceed by induction on i. Notice that the graph G_i is connected and that H_i is an endblock of G_i; moreover, by the BFS algorithm, if $i > 1$, there is only one cutpoint c of G belonging to H_i and appearing in s before H_i. We will denote that cutpoint as $c(i)$. Notice that $c(i)$ is a cutpoint of G_i. All the blocks between $c(i)$ and H_i containing $c(i)$ are endblocks of G_i and are consecutive in s. For each such block H_j, it holds $c(j) = c(i)$.

For each cutpoint c of G, there is only one block containing c and appearing before c in s. We will denote that block by H^c.

We will label the cutpoints of G as A or B, according to some rules, in decreasing order with respect to s. As s was obtained by a BFS of $bc(G)$, if the cutpoint c is being labeled, all the other cutpoints of the blocks containing c and different from H^c are already labeled. The cutpoint c will be labeled B if it belongs to at least two blocks, different from H^c, such that each of them either has at least four cutpoints or has exactly three cutpoints and one of them is already labeled B. The cutpoint c will be labeled A otherwise.

We will show by induction on i that we can find a B_0-VPG representation of G_i such that if c is a cutpoint of G that is a vertex of G_i, then it corresponds to the farthest North, South, East or West line of the line or cross representation of the clique H^c and, moreover, if c is labeled B, then it corresponds to the farthest North and South, or East and West (simultaneously) line of the line or cross representation of the clique H^c.

Claim. If G is $\mathcal{F}$-free, then the following conditions hold: (i) no block of G has five (or more) cutpoints; (ii) a cutpoint c labeled B belongs to exactly two blocks, different from H^c, such that each of them either has at least four cutpoints or has exactly three cutpoints and one of them (different from c) is labeled B; (iii) if a cutpoint c is labeled B, then H^c has at most three cutpoints; and (iv) no block of G having at least three cutpoints is H^{c_1} and H^{c_2} for two cutpoints c_1 and c_2 labeled B.

Proof of the claim. Condition (i) holds since G is N_5-free. Let us assume from now on that (i) is satisfied.

Suppose, for the sake of contradiction, that one of conditions (ii), (iii) or (iv) does not hold. We will prove, by induction on the number of cutpoints labeled B on $bc(G)$, that G contains a member of $\mathcal{F}$ as an induced subgraph.

If there is only one vertex v labeled B, then the conditions that should fail are (ii) or (iii). By the labeling rules and since v is the only vertex labeled B, it belongs to at least two blocks, different from H^v, such that each of them has four cutpoints. Either if the number of such blocks

is at least three or if H^v has four cutpoints, then G contains the second graph in Figure 2 as an induced subgraph.

Suppose that the number of vertices labeled B is greater than one, and let v be the first vertex labeled B in the BFS sequence s (i.e., the one with higher index in s).

By the labeling rules and since v is the first vertex labeled B, it belongs to at least two blocks, different from H^v, such that each of them has four cutpoints. Either if the number of such blocks is at least three or if H^v has four cutpoints, G contains the second graph in Figure 2 as an induced subgraph. Assume then that v belongs to exactly two blocks, different from H^v, such that each of them has four cutpoints, and that H^v has at most three cutpoints.

If H^v has three cutpoints, let w be another cutpoint of G such that $H^v = H^w$. If w is labeled B, since s is a BFS order and v is the first vertex labeled B, w belongs to at least two blocks, different from H^v, such that each of them has four cutpoints. Then G contains the third graph in Figure 2 as an induced subgraph. If w is labeled A, conditions (ii), (iii) and (iv) are "locally" satisfied by v, w, and H^v. We can replace v and all the connected components of $G - v$, except the one containing $H^v - v$, by four vertices v_1, v_2, v'_1 and v'_2 by making v_1 and v_2 adjacent to each other and to $H^v - v$, v'_1 adjacent just to v_1, and v'_2 adjacent just to v_2. Call G' the obtained graph. Now the block $H' = H^v - v \cup \{v_1, v_2\}$ of G' has four cutpoints (all of them labeled A), so the label of every cutpoint placed before v in s remains unchanged in a labeling of $bc(G')$, and the condition among (ii), (iii) and (iv) that was violated in G is still violated in G'. Since all cutpoints of H' are labeled A, G' has one less cutpoint labeled B than G. By the inductive hypothesis, G' contains a graph F of $\mathcal{F}$ as an induced subgraph. Notice that $G' - \{v_1, v'_1\}$ and $G' - \{v_2, v'_2\}$ are isomorphic to an induced subgraph of G. So, since F is connected, if F does not contain one of $\{v_1, v_2\}$, then G contains F as an induced subgraph. If F contains v_1 and v_2, by Proposition 2, F contains $H' \cup \{v'_1, v'_2\}$, and H' is an almost endblock of F. Let F' be the graph obtained from F by applying the procedure given in the definition of $\mathcal{F}$ to the vertices v_1 and v_2. Then F' belongs to $\mathcal{F}$ and F' is an induced subgraph of G.

If H^v has two cutpoints, conditions (ii), (iii) and (iv) are "locally" satisfied by v and H^v, and the label of the other cutpoint of H^v does not depend on the block H^v. We can delete from G all the connected components of $G - v$, except the one containing $H^v - v$, and call G' the obtained graph. The block H is now an endblock of G', the label of every cutpoint placed before v in s remains unchanged in a labeling of $bc(G')$, and the condition among (ii), (iii) and (iv) that was violated in G is still violated in G'. Moreover, v is no longer a cutpoint in G', so G' has one less cutpoint labeled B than G. By the inductive hypothesis, G' contains a graph F of $\mathcal{F}$ as an induced subgraph. Since G' is an induced subgraph of G, so is F. $\diamondsuit$

As a block H is H^c for all but at most one of its cutpoints c, item (iii) of the previous claim implies that no block has four cutpoints such that two of them labeled B, and item (iv) of the previous claim implies that no block has three cutpoints labeled B.

Since, by item (i), no block has five or more cutpoints, the possible label multisets for the blocks of G are $\{A\}$, $\{B\}$, $\{A, A\}$, $\{A, B\}$, $\{B, B\}$, $\{A, A, A\}$, $\{A, A, B\}$, $\{A, B, B\}$, $\{A, A, A, A\}$ and $\{A, A, A, B\}$.

Let $i = 1$, so G_i has only one block H_1. Note that H_1 is H^c for every cutpoint c of G belonging to H_1. So, considering the label multiset of the vertices of H_1, the cases $\{A, A, A, B\}$ and $\{A, B, B\}$ cannot arise (by items (iii) and (iv) of the claim, respectively). In the cases $\{A\}$, $\{B\}$, and $\{A, A\}$, the block can be represented either as a line clique or as a cross clique, satisfying the conditions. In the cases $\{A, B\}$ and $\{B, B\}$, the block can be represented as a cross clique where one of the labeled vertices is the farthest North and South line, and the other one is the farthest East and West line. In the cases $\{A, A, A\}$ and $\{A, A, B\}$, the block can be represented as a cross clique where one of the labeled vertices (the vertex labeled B in the second case) is the farthest North and South line, and the other two are the farthest East, respectively West, line. In the case $\{A, A, A, A\}$, the block can be represented as a cross clique where each labeled vertex corresponds to the farthest North, South, East or West line.

We will proceed now by induction. Let $i > 1$, and let $v := c(i)$, the only cutpoint of H_i appearing in s before H_i. Let $H_j, H_{j+1}, \ldots, H_i$ be the blocks between v and H_i containing v (it

can be $j = i$). As noticed above, $H_j, H_{j+1}, \ldots, H_i$ are endblocks, and since the first element of s is a block, $j > 1$. In particular, $H^v \subseteq G_{j-1}$. Notice also that for $j \leq k \leq i$ and for every cutpoint c of G, different from v, that belongs to H_k, it holds $H_k = H^c$.

We know by the inductive hypothesis that there is a B_0-VPG representation of G_{j-1} such that each cutpoint c of G that belongs to G_{j-1} corresponds to the farthest North, South, East or West line of the line or cross representation of the clique H^c and, moreover, if c is labeled B, then it corresponds to the farthest North and South, or East and West (simultaneously) line of the line or cross representation of the clique H^c.

We will show that, possibly refining the grid, we can extend this representation to a representation of G_i with the desired properties.

We will consider the possible cases for the label of v and the remaining labeled vertices of $H_j, \ldots, H_i$.

Case 1: v is labeled B.

Without loss of generality, assume that vertex v corresponds to the farthest North and South line of the representation of H^v, say P_v. As H^v is the only clique of G_{j-1} containing v, P_v has two segments P_v^N and P_v^S that do not intersect any other path in G_{j-1}, and each of them contains an endpoint of P_v.

Since v is labeled B, we have that the possible multisets for the blocks $H_j, \ldots, H_i$ are $\{B\}$, $\{A, B\}$, $\{B, B\}$, $\{A, A, B\}$, $\{A, B, B\}$, and $\{A, A, A, B\}$. By the item (ii) of the claim, at most two of them have labels $\{A, A, A, B\}$ or $\{A, B, B\}$ (there are exactly two such blocks in G, but some of them may have index greater than i). We will assign to each block a segment of P_v^N or P_v^S, in such a way that the blocks having labels $\{A, A, A, B\}$ or $\{A, B, B\}$ receive the segments of P_v^N, respectively P_v^S, that contain an endpoint of P_v. It is easy to see that we can extend the representation to a B_0-VPG representation of H satisfying the required properties:

- in the case of labels $\{B\}$, we add the remaining vertices in a line clique on the assigned segment;
- in the case of labels $\{A, B\}$ or $\{B, B\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that the other labeled vertex corresponds to the farthest East and West line of the clique;
- in the case of labels $\{A, A, B\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that the other two labeled vertices correspond to the farthest East, respectively West, line of the clique;
- in the case of labels $\{A, B, B\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that the other vertex labeled B corresponds to the farthest East and West line of the clique, and the third labeled vertex corresponds to the farthest North line if the segment assigned contains the North endpoint of P_v, and to the farthest South line, otherwise;
- finally, in the case of labels $\{A, A, A, B\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that two of the other labeled vertices correspond to the farthest East, respectively West, line of the clique, and the third labeled vertex corresponds to the farthest North line if the segment assigned contains the North endpoint of P_v, and to the farthest South line, otherwise.

For a scheme, see Figure 4(a).

Case 2: v is labeled A.

Without loss of generality, assume that vertex v corresponds to the farthest North line of the representation of H^v, say P_v. As H^v is the only clique of G_{j-1} containing v, P_v has a segment P_v^N that does not intersect any other path, and contains the North endpoint of P_v.

Since v is labeled A, the possible multisets for the blocks $H_j, \ldots, H_i$ are $\{A\}$, $\{A, A\}$, $\{A, B\}$, $\{A, A, A\}$, $\{A, A, B\}$, $\{A, A, A, A\}$. Notice that, since for $j \leq k \leq i$ and for every cutpoint c of G,
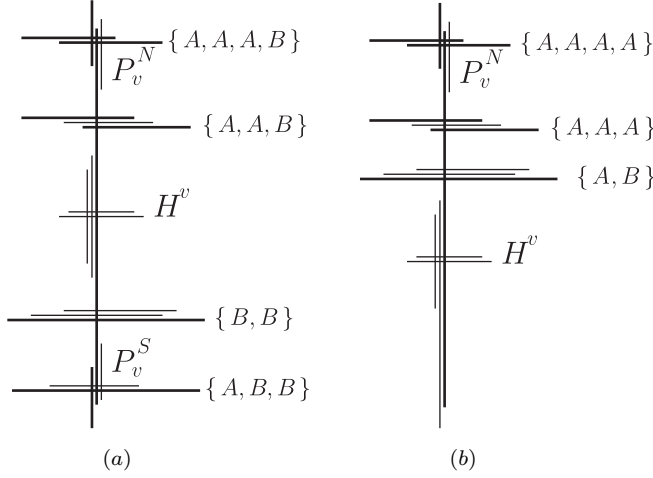
Fig. 4 Scheme for the extension of a representation of G_{j-1} to G_i. The cutpoints are represented by bold lines.

different from v, that belongs to H_k, it holds $H_k = H^c$, the multisets $\{A, A, A, B\}$ and $\{A, B, B\}$ cannot arise (by items (iii) and (iv) of the claim, respectively).

By the labeling rules, at most one block in $H_j, \ldots, H_i$ has labels $\{A, A, A, A\}$ or $\{A, A, B\}$. We will assign to each block a segment of P_v^N, in such a way that the block having labels $\{A, A, A, A\}$ or $\{A, A, B\}$ receives the segment of P_v^N that contains the North endpoint of P_v. It is easy to see that we can extend the representation to a B_0-VPG representation of H satisfying the required properties:

- in the case of labels $\{A\}$, we add the remaining vertices in a line clique on the assigned segment;
- in the case of labels $\{A, A\}$ or $\{A, B\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that the other labeled vertex corresponds to the farthest East and West line of the clique;
- in the case of labels $\{A, A, A\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that the other two labeled vertices correspond to the farthest East, respectively West, line of the clique;
- in the case of labels $\{A, A, B\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that the vertex labeled B corresponds to the farthest East and West line of the clique, and the third labeled vertex corresponds to the farthest North line of the clique;
- finally, in the case of labels $\{A, A, A, B\}$, we add the remaining vertices in a cross clique on the assigned segment, in such a way that two of the other labeled vertices correspond to the farthest East, respectively West, line of the clique, and the third labeled vertex corresponds to the farthest North line of the clique.

For a scheme, see Figure 4(b).

The minimality holds by the equivalence of B_0-VPG and $\mathcal{F}$-free within block graphs, and Corollary 4. □

Corollary 10 *Let G be a chordal diamond-free VPG graph. Then G is a B_0-VPG graph if and only if G is $\mathcal{F}$-free.*

Proof It follows directly by the fact that block graphs are connected chordal diamond-free graphs. □

4 Conclusion

In this paper we consider B_0-VPG graphs, that is, intersection graphs of paths on a grid such that each path is either a horizontal path or a vertical path on the grid. We characterize whether a block graph is a B_0-VPG graph in terms of minimal forbidden induced subgraphs.

The proof of Theorem 9 (i.e., the labeling process and the conditions of the claim) provides an alternative recognition and representation algorithm for B_0-VPG graphs in the class of block graphs. This algorithm is a certifying algorithm, since if the answer is negative it provides a minimal forbidden induced subgraph.

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References

1. A. Asinowski, E. Cohen, M.C. Golumbic, V. Limouzy, M. Lipshteyn, and M. Stern. Vertex intersection graphs of paths on a grid. *Journal of Graph Algorithms and Applications*, 16(2):129–150, 2012.
2. J.A. Bondy and U.S.R. Murty. *Graph Theory*. Springer, New York, 2007.
3. M.L. Brady and M. Sarrafzadeh. Stretching a knock-knee layout of multilayer wiring. *IEEE Transactions on Computers*, 39:148–151, 1990.
4. S. Chaplick, E. Cohen, and J. Stacho. Recognizing some subclasses of vertex intersection graphs of 0-bend paths in a grid. *Lecture Notes in Computer Science*, 6986:319–330, 2011.
5. M.C. Golumbic and B. Ries. On the intersection graphs of orthogonal line segments in the plane: characterizations of some subclasses of chordal graphs. *Graphs and Combinatorics*, 29:499–517, 2013.
6. F. Harary and G. Prins. The block-cutpoint-tree of a graph. *Publicationes Mathematicae Debrecen*, 13:103–107, 1966.
7. J. Kratochvíl and J. Matoušek. Intersection graphs of segments. *Journal of Combinatorial Theory. Series B*, 62:289–315, 1994.
8. P. Molitor. A survey on wiring. *EIK Journal of Information Processing and Cybernetics*, 27:3–19, 1991.
9. F.W. Sinden. Topology of thin film RC circuits. *Bell System Technical Journal*, 45:1639–1662, 1966.