COMPUTATIONAL COMPLEXITY OF CLASSICAL PROBLEMS FOR HEREDITARY CLIQUE-HELLY GRAPHS

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Abstract

A graph is clique-Helly when its cliques satisfy the Helly property. A graph is hereditary clique-Helly when every induced subgraph of it is clique-Helly. The decision problems associated to the stability, chromatic, clique and clique-covering numbers are NP-complete for clique-Helly graphs. In this note, we analyze the complexity of these problems for hereditary clique-Helly graphs. Some of them can be deduced easily by known results. We prove that the clique-covering problem remains NP-complete for hereditary clique-Helly graphs. Furthermore, the decision problems associated to the clique-transversal and the clique-independence numbers are analyzed too. We prove that they remain NP-complete for a smaller class: hereditary clique-Helly split graphs.

Keywords: computational complexity; hereditary clique-Helly graphs; split graphs.

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1. Introduction

All graphs in this paper are finite, without loops or multiple edges. For a graph G we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively.

A graph is complete if every pair of vertices is connected by an edge. A complete in a graph G is a subset of pairwise adjacent vertices of G. A clique in a graph is a complete maximal under inclusion. The clique number of a graph G is the cardinality of a maximum clique of G and is denoted by $\omega(G)$.

The chromatic number $\chi(G)$ of a graph G is the smallest number of colours that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same colour.

A clique cover of a graph G is a subset of cliques covering all the vertices of G. A clique-transversal is a set of vertices intersecting all the cliques of G. The clique-covering number $k(G)$ and the clique-transversal number $\tau_C(G)$ are the cardinalities of a minimum clique cover and a minimum clique-transversal of G, respectively.

A stable set in a graph G is a subset of pairwise non-adjacent vertices of G. A clique-independent set is a subset of pairwise disjoint cliques of G. The stability number $\alpha(G)$ and the clique-independence number $\alpha_C(G)$ are the cardinalities of a maximum stable set and a maximum clique-independent set of G, respectively.

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. The clique graph $K(G)$ of G is the intersection graph of the cliques of G.

A family S of subsets satisfies the Helly property when every subfamily of S consisting of pairwise intersecting subsets has a common element. A graph is clique-Helly (*CH*) when its cliques satisfy the Helly property. A graph G is hereditary clique-Helly (*HCH*) when H is clique-Helly for every induced subgraph H of G. These graphs have been characterized in [Pr93] as the graphs which contains none of the four graphs in Figure 1 as an induced subgraph. This characterization leads to a polynomial time recognition algorithm for hereditary clique-Helly graphs.

An interesting survey on clique-Helly and hereditary clique-Helly graphs appears in [Fa02].

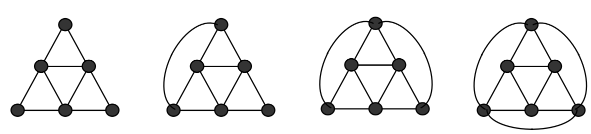

Figure 1. Hajös graphs

A graph is split if its vertices can be partitioned into a clique and a stable set.

The neighborhood of a vertex v in a graph G is the set $N(v)$ consisting of all the vertices that are adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A vertex v of G is called simplicial when $N[v]$ is a complete of G, and universal when $N[v]$=$V(G)$.

It is easy to see that the decision problems associated to the stability, chromatic, clique and clique-covering numbers are NP-complete for clique-Helly graphs. The reduction is trivial: we have to add a universal vertex to the general graph G in order to generate a clique-Helly graph G^+.

However, $\omega(G)$ can be obtained in polynomial time for *HCH* graphs. The number of cliques is bounded by the number of edges [Pr93] and all the cliques can be generated in $O(nmk)$, where m is the number of edges, n the number of vertices and k the number of cliques of the graph [TIAS77].

The stable set and the colorability problems remain NP-complete for *HCH* graphs. These results are direct corollaries of the NP-completeness of these problems for triangle-free graphs [Pol74],[MP96]. For triangle-free graphs, a subclass of *HCH* graphs, the clique-covering number can be obtained in polynomial time [GJ79].

So, the following question arises naturally: what happens with the complexity of the clique-cover problem for hereditary clique-Helly graphs?

The decision problems associated to the problems of finding the clique-independence number and the clique-transversal number are NP-complete [CFT93] and NP-hard [EGT92], respectively. This last problem is not known to be in NP, in fact the problem of determining if a subset of vertices is a clique-transversal is NP-hard [DLS02].

The clique-transversal problem is NP-complete for *HCH* graphs. Again, this result is a consequence of the NP-completeness of this problem for triangle-free graphs. In this class of graphs, the clique-transversal problem is equivalent to vertex cover, and vertex cover is NP-complete for triangle-free graphs [Pol74]. Remember that in this case the problem is in NP for the property of *HCH* graphs above mentioned. This problem remains NP-complete for split graphs [GP00].

However, the clique-independence number can be obtained in polynomial time for triangle-free graphs, because it is equivalent in this case to maximum matching. This problem is NP-complete for split graphs [GP00] but, to our knowledge, it was not known its complexity for clique-Helly graphs.

Again, the following question appears naturally: what happens with the complexity of the clique-independence problem for hereditary clique-Helly graphs?

In this note, we prove that clique-cover and clique-independence problems remain NP-complete for *HCH* graphs. Additionally, it is proved that clique-transversal and clique-independence problems remain NP-complete for a smaller class: the intersection between *HCH* and split graphs.

2. Preliminaries

There are some relations between the parameters defined in the introduction in a graph G and its clique graph $K(G)$.

Theorem 2.1 *Let G be a graph. Then:*

(i) $\alpha_C(G) = \alpha(K(G))$.

(ii) *If G is a clique-Helly graph then* $\tau_C(G) = k(K(G))$.

Proof: (i) It follows from the fact that independent cliques of G correspond to non adjacent vertices in $K(G)$, and conversely, non adjacent vertices in $K(G)$ correspond to independent cliques in G.

(ii) Let $v_1,\ldots,v_{\tau_C(G)}$ be a clique-transversal set of G. For each i, analyze the vertices in $K(G)$ corresponding to the cliques in G that contain the vertex v_i. They form a complete of $K(G)$. This complete must be included in some clique L_i of $K(G)$. Observe that these cliques L_i ($i=1,\ldots,\tau_C(G)$) are not all necessarily different. Let us see that these at most $\tau_C(G)$ cliques are a clique cover of $K(G)$. Let w be a vertex of $K(G)$. Then w corresponds to some clique M_w of G. As the set $v_1,\ldots,v_{\tau_C(G)}$ intersects all the cliques of G, there is some vertex v_j that belongs to M_w. This means that the corresponding vertex of M_w in $K(G)$ belongs to the clique L_j, i.e, $w \in L_j$. Then, the size of the minimum clique cover of $K(G)$ is at most the size of this clique cover which is at most $\tau_C(G)$.

All we need to prove is that if G is clique-Helly, then $\tau_C(G) \leq k(K(G))$. By the Helly property, each clique L of $K(G)$ has an associated vertex v_L in G such that v_L belongs to all the cliques of G corresponding to the vertices of L in $K(G)$.

Let $L_1,\ldots,L_{k(K(G))}$ be a clique cover of $K(G)$. Let $v_{L_1},\ldots,v_{L_{k(K(G))}}$ be the vertices in G associated to those $k(K(G))$ cliques. Let us see that they form a clique-transversal set of G. Let M be a clique of G and w_M its corresponding vertex in $K(G)$. Then there is an index j such that w_M belongs to the clique L_j in $K(G)$. It follows that v_{L_j} belongs to M in G. □

Let $M_1,\ldots,M_k$ and $v_1,\ldots,v_n$ be the cliques and vertices of a graph G, respectively. A clique matrix $A_G \in \mathbb{R}^{k\times n}$ of G is a 0-1 matrix whose entry a_{ij} is 1 if $v_j \in M_i$, and 0, otherwise.

Another characterization of *HCH* graphs is the following [Pr93]: a graph G is *HCH* if and only if A_G does not contain a vertex-edge incidence matrix of a triangle as a submatrix.

Let $M_1,\ldots,M_k$ and $v_1,\ldots,v_n$ be the cliques and vertices of a graph G, respectively. Define the graph $H(G)$ where $V(H(G)) = \{q_1,\ldots,q_k,w_1,\ldots,w_n\}$, each q_i corresponds to the clique M_i of G, and each w_j corresponds to the vertex v_j of G. The edges of $H(G)$ are the following: the vertices $q_1,\ldots,q_k$ induce the graph $K(G)$, the vertices $w_1,\ldots,w_n$ are a stable set and w_j is adjacent to q_i if and only if v_j belongs to the clique M_i in G.

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times k}$ be two matrices. We define the matrix $A\,|\,B \in \mathbb{R}^{n\times(m+k)}$ as $(A\,|\,B)(i,j) = A(i,j)$ for i=1,…,n, j=1,…,m and $(A\,|\,B)(i,m+j) = B(i,j)$ for i=1,…,n, j=1,…,k. Let I_n be the $n\times n$ identity matrix.

Theorem 2.2 [Ham68] *Let G be a clique-Helly graph and H(G) as it is defined above. Then the cliques of H(G) are* $N[w_i]$ *for each i,* w_i *is a simplicial vertex of H(G) for every i, and K(H(G)) = G.*

Corollary 2.1 *Let G be a clique-Helly graph,* $|V(G)| = n$. *Then* $A_{H(G)} = A_G^t\,|\,I_n$.

Proof: It follows directly from the fact that $N[w_i]$ (i=1,…,n) are the cliques of $H(G)$ and each clique contains the vertex w_i and the vertices q_j whose corresponding cliques M_j contain the vertex v_i in G. □

This corollary leads us to prove the following result:

Theorem 2.3 *Let G be an HCH graph. Then H(G) is HCH.*

Proof: Let G be an *HCH* graph, $|V(G)| = n$. By Corollary 2.1, $A_{H(G)} = A_G^t \mid I_n$. Let us suppose that $A_{H(G)}$ contains a vertex-edge incidence matrix of a triangle as a submatrix. Since it has two 1's in each column, it must be a submatrix of A_G^t, but then A_G contains a vertex-edge incidence matrix of a triangle as a submatrix, which is a contradiction. □

3. Clique cover

The decision problem associated to the problem of finding the clique-covering number of a graph is the following:

CLIQUE COVER
INSTANCE: A graph $G = (V,E)$, a positive integer $K \le |V|$.
QUESTION: Are there $k \le K$ cliques of G covering all the vertices of G ?

To prove that CLIQUE COVER is NP-complete for *HCH* graphs, we will use that the following problem is NP-complete [GJ79]:

EXACT COVER BY 3-SETS (X3C)
INSTANCE: A set X such that $|X|=3q$ and a collection C of 3-element subsets of X.
QUESTION: Does C contain an exact cover (by q sets) of X ?

Theorem 3.1 *The problem* CLIQUE COVER *is NP-complete for HCH graphs.*

Proof: The transformation from X3C to CLIQUE COVER on *HCH* graphs is based on the transformation given in [GJ79] from X3C to PARTITION INTO TRIANGLES and is the following: let the set X with $|X|=3q$ and the collection C of 3-element subsets of X be an arbitrary instance of X3C. We will construct an *HCH* graph $G=(V,E)$, with $|V|=3q'$, such that G has a clique cover of size at most q' if and only if C contains an exact cover of X.

We will replace each subset $c_i = \{x_i, y_i, z_i\}$ in C by the graph of Figure 2. Let E_i be the set of 18 edges of the graph corresponding to $\{x_i, y_i, z_i\}$.

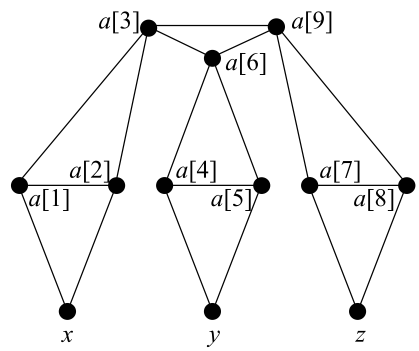

Figure 2. Local replacement for $c=\{x,y,z\}$ in C for transforming X3C to CLIQUE COVER.

Thus $G=(V,E)$ is defined by

$$V = X \cup \bigcup_{i=1}^{|C|} \{a_i[j] : 1 \le j \le 9\},\ \ E = \bigcup_{i=1}^{|C|} E_i$$

It is easy to see that G does not contain any graph of Figure 1 as an induced subgraph, thus G is an *HCH* graph, $|V| = |X| + 9|C|$ ($q' = q + 3|C|$) and the transformation can be made in polynomial time. Figure 3 shows an example of this transformation from an instance of X3C to an instance of CLIQUE COVER.

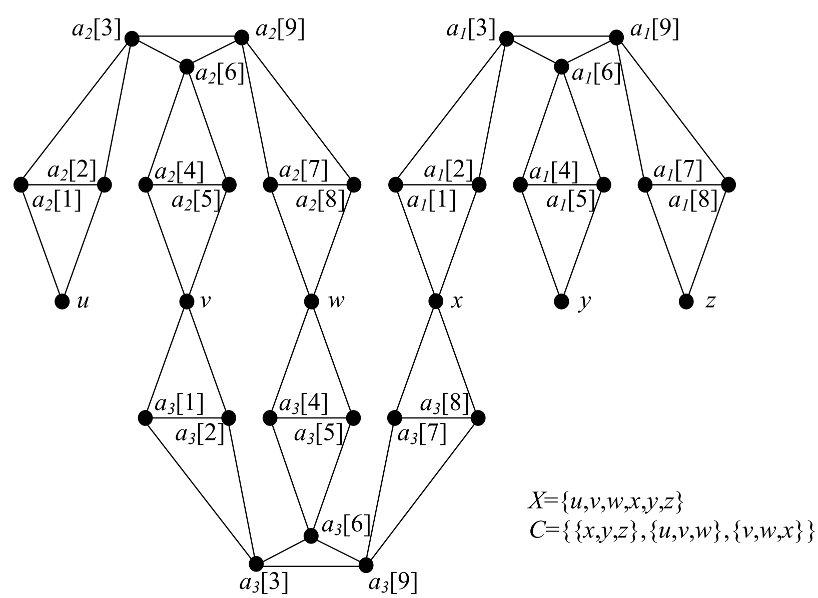

Figure 3. Transformation from an instance of X3C to an instance of CLIQUE COVER.

Let us suppose that C contains an exact cover of X, then we construct a clique cover of G of size q', by taking for each $1 \le i \le |C|$

$$\{a_i[1], a_i[2], x_i\}, \{a_i[4], a_i[5], y_i\}, \{a_i[7], a_i[8], z_i\}, \{a_i[3], a_i[6], a_i[9]\},$$

whenever $c_i = \{x_i, y_i, z_i\}$ is in the exact cover and

$$\{a_i[1], a_i[2], a_i[3]\}, \{a_i[4], a_i[5], a_i[6]\}, \{a_i[7], a_i[8], a_i[9]\},$$

otherwise.

Let us now suppose that *G* has a clique cover of size at most *q*'. Since the cliques of *G* are triangles, the number of cliques in the clique cover must be *q*' and each vertex of *G* must be covered exactly once.

In the graph of Figure 2, the only two ways of covering by triangles each vertex $a_i[j]$ (*j*=1,…,9) exactly once are the above mentioned, covering or not x_i, y_i and z_i, respectively. Then the exact cover of *X* is given by choosing those $c_i \in C$ such that $\{a_i[3], a_i[6], a_i[9]\}$ belongs to the clique cover of *G*.

Finally, the membership in NP for the restricted problem follows from that for the general problem. □

4. Clique transversal and clique-independent set

The decision problems associated to the problems of finding the clique-independence number and the clique-transversal number of a graph, respectively, are the following:

CLIQUE-INDEPENDENT SET
INSTANCE: A graph *G* = (*V*,*E*), a positive integer $K \leq |V|$.
QUESTION: Is there a set of *K* or more pairwise disjoint cliques of *G* ?

CLIQUE-TRANSVERSAL
INSTANCE: *G* = (*V*,*E*), a positive integer $K \leq |V|$.
QUESTION: Is there a set of *K* or fewer vertices of *G* intersecting all the cliques of *G* ?

Theorem 4.1 *The problems* CLIQUE-TRANSVERSAL *and* CLIQUE-INDEPENDENT SET *are NP-complete for HCH split graphs.*

Proof: We will show a polynomial time transformation from CLIQUE COVER on *HCH* graphs (by Theorem 3.1 it is NP-complete) to CLIQUE-TRANSVERSAL on *HCH* split graphs.

Define the graph G^+ where $V(G^+) = V(G) \cup \{u\}$, $V(G)$ induces the graph *G* and *u* is a universal vertex. Since for any graph *G* all the cliques of G^+ share the vertex *u*, the graph $K(G^+)$ is complete and thus the graph $H(G^+)$ is a split graph.

Let *G* be an *HCH* graph. As the set of cliques of an *HCH* graph has polynomial size and can be computed in polynomial time, $H(G^+)$ can be built in polynomial time. By Theorem 2.3, since G^+ is an *HCH* graph, $H(G^+)$ is an *HCH* graph. By Theorem 2.2 $K(H(G^+)) = G^+$, and by Theorem 2.1 $k(G) = k(G^+) = \tau_C(H(G^+))$. Finally, the problem of determining if a subset of vertices is a clique-transversal is solvable in polynomial time for *HCH* graphs, and therefore CLIQUE-TRANSVERSAL is NP-complete for *HCH* split graphs.

In a similar way, using the equality $\alpha(G)=\alpha(G^{+})=\alpha_{C}(H(G^{+}))$ instead of $k(G)=k(G^{+})=\tau_{C}(H(G^{+}))$, and the NP-completeness of the STABLE SET problem for *HCH* graphs, CLIQUE-INDEPENDENT SET is NP-complete for *HCH* split graphs. □

Corollary 4.1 *The problem* CLIQUE-INDEPENDENT SET *is NP-complete for HCH graphs.*

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References

[CFT93] Chang, M., Farber, M. & Tuza, Z. (1993). Algorithmic aspects of neighbourhood numbers. SIAM Journal on Discrete Mathematics, 6, 24-29.

[DLS02] Durán, G., Lin, M. & Szwarcfiter, J. (2002). On clique-transversal and clique-independent sets. Annals of Operations Research, 116, 71-77.

[EGT92] Erdös, P., Gallai, T. & Tuza, Z. (1992). Covering the cliques of a graph with vertices. Discrete Mathematics, 108, 279-289.

[Fa02] Farrugia, A. (2002). Clique-Helly graphs and hereditary clique-Helly graphs, a mini-survey. Manuscript.

[GJ79] Garey, M. & Johnson, D. (1979). Computers and Intractability: A Guide to the Theory of NP-Completeness. Freeman and Company, San Francisco.

[GP00] Guruswami, V. & Pandu Rangan, C. (2000). Algorithmic aspects of clique-transversal and clique-independent sets. Discrete Applied Mathematics, 100, 183-202.

[Ham68] Hamelink, R. (1968). A partial characterization of clique graphs. Journal of Combinatorial Theory, Series B, 5, 192-197.

[MP96] Maffray, F. & Preissmann, M. (1996). On the NP-completeness of the *k*-colorability problem for triangle-free graphs. Discrete Mathematics, 162, 313-317.

[Pol74] Poljak, S. (1974). A note on stable sets and colorings of graphs. Commentationes Mathematicae Universitatis Carolinae, 15, 307-309.

[Pr93] Prisner, E. (1993). Hereditary clique-Helly graphs. Journal of Combinatorial Mathematics and Combinatorial Computing, 14, 216-220.

[TIAS77] Tsukiyama, S., Idle, M., Ariyoshi, H. & Shirakawa, Y. (1977). A new algorithm for generating all the maximal independent sets. SIAM Journal on Computing, 6 (3), 505-517.