Partial characterizations of clique-perfect and coordinated graphs: superclasses of triangle-free graphs

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Abstract

A graph is *clique-perfect* if the cardinality of a maximum clique-independent set equals the cardinality of a minimum clique-transversal, for all its induced subgraphs. A graph G is *coordinated* if the chromatic number of the clique graph of H equals the maximum number of cliques of H with a common vertex, for every induced subgraph H of G. Coordinated graphs are a subclass of perfect graphs. The complete lists of minimal forbidden induced subgraphs for the classes of clique-perfect and coordinated graphs are not known, but some partial characterizations have been obtained. In this paper, we characterize clique-perfect and coordinated graphs by minimal forbidden induced subgraphs when the graph is either paw-free or {gem,W_4,bull}-free, two superclasses of triangle-free graphs.

Keywords: Clique-perfect graphs, coordinated graphs, {gem,W_4,bull}-free graphs, paw-free graphs, perfect graphs, triangle-free graphs.

1 Introduction

A graph G is *perfect* if the chromatic number equals the clique number for every induced subgraph H of G. A graph G is perfect if and only if no induced subgraph of G is an odd hole or an odd antihole [7]. This class of graphs can be recognized in polynomial time [6].

A graph G is *clique-Helly* (CH) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (HCH) if H is clique-Helly for every induced subgraph H of G. The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. A graph G is *K-perfect* if $K(G)$ is perfect.

A *paw* is a triangle with a leaf attached to one of its vertices. A *gem* is a graph of five vertices, such that four of them induce a chordless path and the fifth vertex is universal. A *bull* is a triangle with two leafs attached to different vertices of it. A *wheel* W_j is a graph of $j+1$ vertices, such that j of them induce a chordless cycle and the last vertex is universal. We say that a graph is *anticonnected* if its complement is connected. An *anticomponent* of a graph is a connected component of its complement.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* of G, $\tau_C(G)$, and the *clique-independence number* of G, $\alpha_C(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. Clearly, $\alpha_C(G) \geq \tau_C(G)$, for any graph G. A graph G is *clique-perfect* [10] if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G. The only clique-perfect graphs which are minimally imperfect are the odd antiholes of length $6j+3$, for any $j \geq 1$ [4]. The complexity of the recognition problem for clique-perfect graphs is still not known.

A *K-coloring* of a graph G is a coloring of $K(G)$. A *Helly K-complete* of a graph G is a collection of cliques of G with common intersection. The *K-chromatic number* and the *Helly K-clique number* of G, denoted by $F(G)$ and $M(G)$, are the sizes of a minimum K-coloring and a maximum Helly K-complete of G, respectively. It is easy to see that $F(G) \geq M(G)$ for any graph G. A graph G is *C-good* if $F(G) = M(G)$. A graph G is *coordinated* if

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every induced subgraph of G is C-good. Coordinated graphs were defined and studied in [3], where it was proved that they are a subclass of perfect graphs. The recognition problem for coordinated graphs is NP-hard and remains NP-complete when restricted to some subclasses of graphs with $M(G) = 3$ [18].

A class of graphs $\mathcal{C}$ is *hereditary* if for every $G \in \mathcal{C}$, every induced subgraph of G also belongs to $\mathcal{C}$.

Finding the complete lists of minimal forbidden induced subgraphs for the classes of clique-perfect and coordinated graphs turns out to be a difficult task ([1,17]). However, some partial characterizations have been obtained in previous works (see [1,2,5,11]). In this paper, we characterize clique-perfect and coordinated graphs by minimal forbidden induced subgraphs when the graph is either paw-free or {gem,W_4,bull}-free, two superclasses of triangle-free graphs. In particular, we prove that in these cases clique-perfect and coordinated graphs are equivalent to perfect graphs and, in consequence, the only forbidden subgraphs are the odd holes. As a direct corollary, we can deduce polynomial time algorithms to recognize clique-perfect and coordinated graphs when the graph belongs to these classes.

2 Main results

Triangle-free graphs were widely studied in the literature, usually in the context of graph coloring problems (see for example [12,13]). It is easy to see that if a graph is triangle-free then it is perfect if and only if it is clique-perfect, if and only if it is coordinated. We shall extend this result by analyzing two superclasses of triangle-free graphs: paw-free and {gem,W_4,bull}-free graphs.

Paw-free graphs were studied in [14]. In this work we prove that the characterization mentioned above for clique-perfect and coordinated graphs on triangle-free graphs also holds on paw-free graphs.

Lemma 2.1 [14] *Let G be a paw-free graph. If G is not anticonnected then the anticomponents of G are stable sets. If G is connected and anticonnected then G is triangle-free.*

We first prove the following auxiliary results.

Theorem 2.2 *Let G be a paw-free, connected and anticonnected graph. Then G is perfect if and only if G is bipartite.*

Theorem 2.3 *Let G be a paw-free graph. If G is not anticonnected, then G is coordinated.*

Now we can prove the main result for paw-free graphs.

Theorem 2.4 *Let G be a paw-free graph. The following statements are equivalent:*

(i) *G is perfect.*

(ii) *G is clique-perfect.*

(iii) *G is coordinated.*

Proof:

(i) $\Rightarrow$ (ii)) Since the class of paw-free perfect graphs is hereditary, it is enough to see that $\alpha_c(G) = \tau_c(G)$. We can assume that G is connected. If G is anticonnected, then by Theorem 2.2, G is bipartite, and so G is clique-perfect. If G is not anticonnected, then by Lemma 2.1, G has $A_1, \ldots, A_s$ anticomponents with A_i being an stable set, for all $1 \leq i \leq s$. Without loss of generality, we can assume that $|A_1| \leq |A_i|$ $(2 \leq i \leq s)$. Denote $a = |A_1|$. Every clique of G is composed by exactly one vertex of each A_i. Let $v_1^i, \ldots, v_{|A_i|}^i$ be an enumeration of the vertices of A_i (for $1 \leq i \leq s$). For each j $(1 \leq j \leq a)$, let $K_j = \{v_j^1, \ldots, v_j^s\}$. Clearly, K_j is a clique and for $1 \leq i < j \leq a$, $K_j \cap K_i = \emptyset$. Therefore, $K_1, \ldots, K_a$ is a clique-independent set, which implies that $\alpha_c(G) \geq a$. On the other hand, since every clique has a vertex of A_1, then A_1 is a clique-transversal of G. Therefore $\tau_c(G) \leq a$. So, $a \leq \alpha_c(G) \leq \tau_c(G) \leq a$, and hence $\alpha_c(G) = \tau_c(G)$.

(ii) $\Rightarrow$ (iii)) We can assume that G is connected. If G is not anticonnected, then by Theorem 2.3, G is coordinated. If G is anticonnected, then by Lemma 2.1, G has no triangles and therefore G does not have odd antiholes with length greater than 5. On the other hand, since odd holes are not clique-perfect, G has no odd holes. We conclude that G is perfect. Let G' be an induced subgraph of G. To see that G' is C-good, it is enough to prove that every connected component of G' is C-good. Let H be a connected component of G'. If H is not anticonnected, then by Theorem 2.3, H is coordinated; in particular it is C-good. If H is anticonnected, since it is also connected and perfect, by Theorem 2.2 it follows that H is bipartite. Then H is C-good.

(iii) $\Rightarrow$ (i)) Coordinated graphs are a subclass of perfect graphs. □

Corollary 2.5 *Clique-perfect and coordinated graphs can be recognized in linear time when the graph is paw-free.*

Bull-free graphs were studied in the context of perfect graphs [8,16], and {gem,W_4}-free graphs in the context of clique-perfect graphs [9]. It is not difficult to build examples of {gem,W_4}-free perfect graphs which are neither clique-perfect nor coordinated. So, the equivalence of Theorem 2.4 does not hold on this class. But we can prove the same equivalence if we also forbid bulls.

First we will show that if {gem,W_4,bull}-free graphs are perfect, they are K-perfect. We prove the following auxiliary results.

Theorem 2.6 *If G is a $\{gem,W_4\}$-free graph then $K(G)$ is a $\{gem,W_4\}$-free graph, hence $K(G)$ contains no odd antihole of length greater than 5.*

Let G be a graph. A *K-hole* $Q_1, \ldots, Q_k$ ($k \geq 4$) is a set of cliques of G which induces a hole in $K(G)$ (i.e., $Q_i \cap Q_j \neq \emptyset \Leftrightarrow i = j$ or $i \equiv j \pm 1 \mod k$). An *intersection cycle* of a K-hole $Q_1, \ldots, Q_k$ is a cycle $v_1, \ldots, v_k$ of G such that $v_i \in Q_i \cap Q_{i+1}$ for every $i, 1 \leq i \leq k$. Let C be a cycle of a graph G. An edge (v, w) of C is *improper* if there is a vertex $z \in C$ such that v, w, z is a triangle. An edge of C is *proper* if it is not improper.

Lemma 2.7 *Let G be a perfect $\{gem,W_4,bull\}$-free graph and $C = v_1, \ldots, v_{2k+1}$ ($k \geq 2$) an intersection cycle of a K-hole $Q_1, \ldots, Q_{2k+1}$. Then C contains neither two consecutive improper edges nor two consecutive proper edges.*

Now we can prove that a perfect {gem,W_4,bull}-free graph is K-perfect.

Theorem 2.8 *If G is a perfect $\{gem,W_4,bull\}$-free graph then G is K-perfect.*

Proof: Suppose G is not K-perfect. By Theorem 2.6, $K(G)$ contains no odd antihole of length greater than 5. Therefore, $K(G)$ contains an odd hole, so there is an odd-length intersection cycle $v_1, \ldots, v_{2k+1}$ ($k \geq 2$) in G. Call $e_i = (v_i, v_{i+1})$ for every $i, 1 \leq i \leq 2k + 1$. By Lemma 2.7 we may assume that e_1 is an improper edge and e_2 is a proper edge. By a repeated application of the same lemma (note that the cycle is odd) we obtain that e_{2k+1} is improper and therefore e_1 is proper, which is a contradiction. □

By the characterization of HCH graphs by forbidden subgraphs [15], {gem,W_4,bull}-free graphs are also HCH. It is known that if $\mathcal{C}$ is an hereditary class of K-perfect clique-Helly graphs, every graph in $\mathcal{C}$ is clique-perfect and coordinated [1,5]. So, since {gem,W_4,bull}-free graphs is an hereditary class of graphs, we obtain as a corollary of Theorem 2.8 the following equivalence.

Theorem 2.9 *Let G be a $\{gem,W_4,bull\}$-free graph. Then G is perfect, if and only if G is clique-perfect, if and only if G is coordinated.*

Corollary 2.10 *Clique-perfect and coordinated graphs can be recognized in polynomial time when the graph is $\{gem,W_4,bull\}$-free.*

It remains as an open problem to determine the "biggest" superclass of triangle-free graphs where perfect, clique-perfect and coordinated graphs are equivalent.

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