Clique-perfectness of complements of line graphs

Flavia Bonomo [a,b,1,4] Guillermo Durán [a,c,d,2,3,4]
Martín D. Safe [a,b,e,3,4] Annegret K. Wagler [f,4]

[a] *CONICET, Argentina*

[b] *Depto. de Computación, FCEN, Universidad de Buenos Aires, Argentina*

[c] *Depto. de Matemática, FCEN, Universidad de Buenos Aires, Argentina*

[d] *Depto. de Ingeniería Industrial, FCFM, Universidad de Chile, Chile*

[e] *Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina*

[f] *ISIMA-CNRS, Université Clermont-Ferrand II (Blaise Pascal), France*

Abstract

The *clique-transversal number* $\tau_{\mathrm{c}}(G)$ of a graph G is the minimum size of a set of vertices meeting all the cliques. The *clique-independence number* $\alpha_{\mathrm{c}}(G)$ of G is the maximum size of a collection of vertex-disjoint cliques. A graph is *clique-perfect* if these two numbers are equal for every induced subgraph of G. Unlike perfect graphs, the class of clique-perfect graphs is not closed under graph complementation nor is a characterization by forbidden induced subgraphs known. Nevertheless, partial results in this direction have been obtained. For instance, in [3], a characterization of those line graphs that are clique-perfect is given in terms of minimal forbidden induced subgraphs. Our main result is a characterization of those complements of line graphs that are clique-perfect, also by means of minimal forbidden induced subgraphs. This implies an $O(n^2)$ time algorithm for deciding the clique-perfectness of complements of line graphs and, for those that are clique-perfect, finding α_{c} and τ_{c}.

Keywords: clique-perfect graphs, edge-coloring, line graphs, maximal matchings

1 Introduction

A *clique* is an inclusion-wise maximal set of pairwise adjacent vertices. A graph is called *perfect* if, for each of its induced subgraphs, the size of a largest clique equals the minimum number of colors needed to assign different colors to adjacent vertices. Like perfect graphs, clique-perfect graphs are defined by the equality of two graph parameters. The *clique-transversal number* $\tau_c(G)$ of a graph G is the minimum size of a set of vertices that meets all the cliques of G and the *clique-independence number* $\alpha_c(G)$ of G is the maximum size of a collection of vertex-disjoint cliques of G. Clearly, $\alpha_c(G) \leq \tau_c(G)$ for every graph G. G is said to be *clique-perfect* if $\alpha_c(G') = \tau_c(G')$ for each induced subgraph G' of G [11]. Not all clique-perfect graphs are perfect and not all perfect graphs are clique-perfect, but graphs belonging to certain graph classes are known to be clique-perfect; e.g., comparability graphs [1], distance-hereditary graphs [14], and dually chordal graphs [7]. Unlike perfect graphs, the class of clique-perfect graphs is not closed under graph complementation nor is a complete characterization of clique-perfect graphs by forbidden induced subgraphs known. Nevertheless, partial results in this direction have been obtained; i.e., characterizations of clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph is known to belong to certain graph classes [3,4,5]. For instance, in [3], a characterization of those line graphs that are clique-perfect is given in terms of minimal forbidden induced subgraphs. Our main result is a characterization of clique-perfect graphs within the complements of line graphs by minimal forbidden induced subgraphs. Another open question about clique-perfect graphs is the complexity of the recognition problem. Our characterization implies an $O(n^2)$ time algorithm for deciding the clique-perfectness of complements of line graphs.

In Section 2, we introduce some definitions and a preliminary result on edge-coloring. In Section 3, we present our characterization of those complements of line graphs that are clique-perfect and from this we derive the existence of an algorithm that given G, the complement of a line graph, decides whether G is clique-perfect and, if affirmative, finds $\alpha_c(G)$ and $\tau_c(G)$.

[1] Partially supported by ANPCyT PICT-2007-00518 and PICT-2007-00533 and UBACyT Grants X069 and 20020090300094 (Argentina)

[2] Partially supported by FONDECyT Grant 1080286 and Millennium Science Institute "Complex Engineering Systems" (Chile)

[3] Partially supported by ANPCyT PICT-2007-00518, UBACyT Grant X069, and CONICET PIP 112-200901-00178 (Argentina)

[4] Email addresses: fbonomo@dc.uba.ar (Flavia Bonomo), gduran@dm.uba.ar (Guillermo Durán), msafe@ungs.edu.ar (Martín D. Safe), wagler@isima.fr (Annegret K. Wagler)

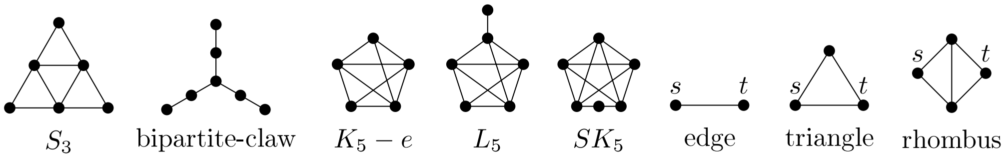

Fig. 1. Some graphs and some two-terminal graphs with terminals s and t

2 Definitions and preliminaries on edge-coloring

Graphs in this work are finite, undirected, without loops, and without multiple edges. Let G be a graph. The vertex set of G is denoted by $V(G)$ and the edge set by $E(G)$. For any set S, $|S|$ denotes its cardinality. The set of neighbors of a vertex v in G is denoted by $N_G(v)$ and $N_G[v]$ denotes $N_G(v) \cup \{v\}$. The *degree* of v is $|N_G(v)|$ and v is *pendant* if its degree is 1. The maximum degree of the vertices of G is denoted by $\Delta(G)$ and the complement of G by $\overline{G}$. We say that G *contains* H if H is a subgraph (induced or not) of G and G *contains an induced* H if H is an induced subgraph of G. Cycles have no repeated vertices (apart from the starting and ending vertices). Let C be a cycle. The *length* of C is the number of edges joining two consecutive vertices of C and C is *odd* if its length is odd. C is *chordless* if there is no edge joining two nonconsecutive vertices of C. A *hole* is a chordless cycle of length at least 5 and an *antihole* is the complement of a hole. The chordless cycle of length n is denoted by C_n and the complete graph on n vertices by K_n. For each $n \geq 5$, the *length* of the antihole $\overline{C}_n$ is n. If H is a graph, the *line graph* $L(H)$ of H has $E(H)$ as vertex set and $e_1, e_2 \in E(H)$ are adjacent in $L(H)$ if and only if e_1 and e_2 share exactly one endpoint. A graph G is a *line graph* if there exists a graph H such that $G = L(H)$; if so, H is called a *root graph* of G. Let G and H be graphs with $V(G) \cap V(H) = \emptyset$. The *disjoint union* $G \cup H$ of G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. A *matching* is a set of pairwise vertex-disjoint edges and a matching is *maximal* if it is inclusion-wise maximal. For some graphs needed hereafter, see Figure 1.

The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors needed to color the edges of G so that no two incident edges receive the same color. Clearly, $\chi'(G) \geq \Delta(G)$. In fact, Vizing [17] proved that for every graph G either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G)+1$. The problem of deciding whether a graph G satisfies $\chi'(G) = \Delta(G)$ is NP-complete even for graphs having only vertices of degree 3 [13]. Our result below is a structural characterization of those graphs that satisfy $\chi' = \Delta$ restricted to graphs not containing a bipartite-claw. Before stating it, we need to introduce the notion of circular concatenation. A *two-terminal* graph is a triple $\Gamma = (G, s, t)$, where s and t

are two vertices of G, called the *terminals* of Γ. For some two-terminal graphs, see Figure 1. If $\Gamma_1 = (G_1, s_1, t_1)$ and $\Gamma_2 = (G_2, s_2, t_2)$ are two-terminal graphs, the *p-concatenation* $\Gamma_1 \,\&_p\, \Gamma_2$ is the two-terminal graph (G, s_1, t_2) where G arises from $G_1 \cup G_2$ by identifying t_1 and s_2 into one vertex u and attaching p pendant vertices adjacent to u. If the two-terminal graph (G, s, t) is such that $N_G[s] \cap N_G[t] = \emptyset$, we define its *p-closure* as the graph that arises by identifying s and t into one vertex u and then attaching p pendant vertices adjacent to u. A *circular concatenation* of the two-terminal graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is the graph that arises as the p_n-closure of $\Gamma_1 \,\&_{p_1}\, \Gamma_2 \,\&_{p_2} \ldots \&_{p_{n-1}}\, \Gamma_n$ for some $p_i \geq 0$, $i = 1, 2, \ldots, n$. Each of the Γ_i's is called a *link* of the circular concatenation. By exploiting the structure of the graphs not containing a bipartite-claw and using results from [12] and [8], we prove the following.

Theorem 2.1 *Let G be a connected graph not having a bipartite-claw as a subgraph. Then, $\chi'(G) = \Delta(G)$ if and only if none of the following holds:*

(i) $\Delta(G) = 2$ *and G is an odd chordless cycle.*

(ii) $\Delta(G) = 3$ *and G is the circular concatenation of a sequence of edges, triangles, and rhombi, where the number of links that are edges equals one plus the number of links that are rhombi.*

(iii) $\Delta(G) = 4$ *and $G = K_5 - e$, K_5, L_5, or SK_5.*

3 Clique-perfectness of complements of line graphs

In [3], clique-perfect graphs were characterized by minimal forbidden induced subgraphs within the class of line graphs: a line graph G is clique-perfect if and only if G contains no induced S_3 and has no odd hole.

Our main result is the following characterization of clique-perfect graphs among complements of line graphs by minimal forbidden induced subgraphs.

Theorem 3.1 *Let G be the complement of a line graph. Then, G is clique-perfect if and only if G contains no induced S_3 and has no antihole $\overline{C}_k$ for every $k \geq 5$ such that k is not a multiple of* 3.

Let G be the complement of the line graph of a graph H. In order to prove Theorem 3.1, we profit from the correspondence between the cliques of G and the maximal matchings of H. We define the *matching-transversal number* $\tau_{\mathrm{m}}(H)$ as the minimum number of edges meeting all the maximal matchings of H and the *matching-independence number* $\alpha_{\mathrm{m}}(H)$ as the maximum number of edge-disjoint maximal matchings of H. We say that a graph H is *matching-perfect* if $\alpha_{\mathrm{m}}(H') = \tau_{\mathrm{m}}(H')$ for every subgraph H' (induced or not) of H.

Clearly, $\alpha_{\rm c}(G) = \alpha_{\rm m}(H)$ and $\tau_{\rm c}(G) = \tau_{\rm m}(H)$. Thus, G is clique-perfect if and only if H is matching-perfect and Theorem 3.1 can be reformulated as follows.

Theorem 3.2 *Let H be a graph. Then, H is matching-perfect if and only if H contains no bipartite-claw and the length of each cycle of H is at most 4 or is a multiple of 3.*

In order to prove Theorem 3.2 it is enough to show that if H is a graph containing no bipartite-claw and the length of each cycle of H is at most 4 or is a multiple of 3 then $\alpha_{\rm m}(H) = \tau_{\rm m}(H)$. The proof splits into two parts according to whether G has some cycle of length at least 5 or not. In both cases, we obtain an upper bound on $\tau_{\rm m}(H)$ and then produce a collection of edge-disjoint maximal matchings of the same size and, therefore, $\alpha_{\rm m}(H) = \tau_{\rm m}(H)$. Most of the times, this collection of maximal matchings arises as the set of color classes of an edge-coloring (via Theorem 2.1) of a tailored subgraph of H.

We now discuss the derivation of the recognition algorithm. The reader unfamiliar with the notions of *treewidth* or *counting monadic second-order (CMS) logic* may consult [9, Ch. 2 & 5]. Since forbidding the bipartite-claw as a subgraph or as a minor are equivalent, graphs containing no bipartite-claw have bounded treewidth [16] and have a linear-time recognition algorithm [2]. Moreover, as "the length of each cycle is at most 4 or is a multiple of 3" can be expressed by CMS logic, it can be evaluated in linear-time over graphs within any graph class of bounded treewidth [6,10]. Thus, matching-perfect graphs can be recognized in linear-time. Finally, if G is the complement of a line graph, it can be decided in $O(|V(G)|^2)$ whether G is clique-perfect by first finding a root graph H of $\overline{G}$ in $O(|V(G)|^2)$ time [15] and then determining whether H is matching-perfect in $O(|V(G)|)$ time. Since for matching-perfect graphs the common value $\alpha_{\rm m} = \tau_{\rm m}$ can be shown to be linear-time computable:

Theorem 3.3 *Deciding whether G, the complement of a line graph, is clique-perfect and, if affirmative, finding $\alpha_{\rm c}(G)$ and $\tau_{\rm c}(G)$, can be done in $O(|V(G)|^2)$.*

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