Partial characterizations of clique-perfect graphs

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Abstract

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. A graph G is *clique-perfect* if the sizes of a minimum clique-transversal and a maximum clique-independent set are equal for every induced subgraph of G. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. In this paper, we present a partial result in this direction, that is, we characterize clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph belongs to a certain class.

Keywords: Claw-free graphs, clique-perfect graphs, diamond-free graphs, hereditary clique-Helly graphs, line graphs, perfect graphs.

1 Introduction

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$, the complement of G. The *line graph* $L(G)$ of G is the intersection graph of the edges of G. A graph F is a *line graph* if there exists a graph H such that $L(H) = F$.

A *complete set* of G is a subset of vertices pairwise adjacent. A *clique* is a complete set not properly contained in any other. We may also use the term clique to refer to the corresponding complete subgraph. A *stable set* in a graph G is a subset of pairwise non-adjacent vertices of G. The *stability number* $\alpha(G)$ is the cardinality of a maximum independent set of G.

A *clique cover* of a graph G is a subset of cliques covering all the vertices of G. The *clique-covering number* $k(G)$ is the cardinality of a minimum clique cover of G. The *chromatic number* of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of G, the *clique number* of G, denoted by $\omega(G)$.

A graph G is *perfect* when $\chi(H) = \omega(H)$ for every induced subgraph H of G. Perfect graphs are very interesting from an algorithmic point of view, see [11]. While determining the clique-covering number, the independence number, the chromatic number and the clique number of a graph are NP-complete problems, they are solvable in polynomial time for perfect graphs [12]. Besides, it has been proved recently that perfect graphs can be characterized by two families of minimal forbidden induced subgraphs [8] and recognized in polynomial time [7]. The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. A graph G is *K-perfect* if $K(G)$ is perfect.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint

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cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_c(G)$ and $\alpha_c(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. It is easy to see that $\tau_c(G) \geq \alpha_c(G)$ for any graph G. A graph G is *clique-perfect* if $\tau_c(H) = \alpha_c(H)$ for every induced subgraph H of G. Clique-perfect graphs have been implicitly studied in [1,2,4,3,5,10,13,14]. The terminology "clique-perfect" has been introduced in [13]. There are two main open problems concerning this class of graphs:

- find all minimal forbidden induced subgraphs for the class of clique-perfect graphs, and
- is there a polynomial time recognition algorithm for this class of graphs?

In this paper, we present a partial result in this direction, that is, we characterize clique-perfect graphs by forbidden subgraphs when the graph belongs to a certain class.

2 Forbidden families for clique-perfect graphs

Let G be a graph and C be a cycle of G not necessarily induced. An edge of C is *non proper* (or *improper*) if it forms a triangle with some vertex of C. An r-*generalized sun*, $r \geq 3$, is a graph G whose vertex set can be partitioned into two sets: a cycle C of r vertices, with all its non proper edges $\{e_j\}_{j \in J}$ (J is permitted be an empty set) and a stable set $U = \{u_j\}_{j \in J}$, such that for each $j \in J$, u_j is adjacent only to the endpoints of e_j. A hole is a chordless cycle of length at least 4. An antihole is the complement of a hole. A hole or antihole is said to be *odd* if its cardinality is odd. An r-generalized sun is said to be *odd* if r is odd. Clearly odd holes are odd generalized suns.

Theorem 2.1 [3] *Odd generalized suns and antiholes of length* $t = 1, 2$ *mod* 3 $(t \geq 5)$ *are not clique-perfect.*

Unfortunately, odd generalized suns are not necessary minimal (with respect to taking induced subgraphs) and besides there are other minimal non-clique-perfect graphs, for example the following family of graphs. Define the graph S_k, $k \geq 2$, as follows: $V(S_k) = \{v_1, \ldots, v_{2k}, v, v', w, w'\}$ where $v_1, \ldots, v_{2k}$ induce a path, v is adjacent to v', v_1, v_2 and v_{2k}; v' is adjacent to v, v_1, v_{2k-1} and v_{2k}; w is adjacent only to v_1 and v_2; w' is adjacent only to v_{2k-1} and v_{2k}.

At this time we do not know whether the list of all such graphs has a nice description. However, if we restrict our attention to certain classes of graphs (that can be described by forbidding certain induced subgraphs), we can describe all the forbidden induced subgraphs, and this is our main result.

3 Partial characterizations

Let us call a class of graphs $\mathcal{C}$ *hereditary* if for every $G \in \mathcal{C}$, every induced subgraph of G also belongs to $\mathcal{C}$. We say that a graph is *interesting* if no induced subgraph of it is an odd generalized sun or an antihole of length greater than 5 and equal to $1, 2$ *mod* 3.

A family of sets S is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element. A graph is *clique-Helly* (CH) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (HCH) if H is clique-Helly for every induced subgraph H of G. Hereditary clique-Helly graphs are of particular interest because in this case it follows from [3] that if $K(H)$ is perfect for every induced subgraph H of G, then G is clique-perfect (the converse is not necessarily true). On the other hand, the class of hereditary clique-Helly graphs can be easily characterized by forbidden induced subgraphs [15]. The following is a useful fact about hereditary clique-Helly graphs:

Proposition 3.1 *Let $\mathcal{L}$ be a hereditary graph class, which is HCH and such that every interesting graph in $\mathcal{L}$ is K-perfect. Then every interesting graph in $\mathcal{L}$ is clique-perfect.*

Proof. Let G be an interesting graph in $\mathcal{L}$. Let H be an induced subgraph of G. Since $\mathcal{L}$ is hereditary, H is an interesting graph in $\mathcal{L}$, so it is K-perfect. Since $\mathcal{L}$ is a HCH class, H is clique-Helly and then $\alpha_C(H) = \alpha(K(H)) = k(K(H)) = \tau_C(H)$ [3], and the result follows. $\square$

A *claw* is the graph $K_{1,3}$, and a *diamond* is the graph $K_4 - \{e\}$, it follows from [15] that diamond-free graphs are HCH. Our main result is the following:

Theorem 3.2 *Let G be a graph which is either diamond-free, or a line graph, or HCH and claw-free. Then G is clique-perfect if and only if it is interesting.*

The proof of Theorem 3.2 in the diamond-free case is rather simple, we show:

Theorem 3.3 *If G is diamond-free, the following are equivalent:*

(i) *G is interesting*

(ii) *G contains no odd generalized sun*

(iii) *G is clique-perfect.*

Let us prove that (ii) implies (iii). First we can show that $K(G)$ is Berge, and therefore perfect. By [6], G being diamond-free implies that $K(G)$ is diamond free, and hence $K(G)$ contains no antihole of length at least 7. Suppose $K(G)$ contains an odd hole $k_1k_2\dots k_{2n+1}$, where $k_1,\dots,k_{2n+1}$ are cliques of G. Then G contains an odd cycle $v_1v_2\dots v_{2n+1}$, where v_i belongs to $k_i \cap k_{i+1}$ and no other k_j. Since G contains no odd generalized suns, we may assume that some edge of this cycle, say, (v_1, v_2) is in a triangle with another vertex of the cycle, say v_m. Now v_1, v_2 both belong to k_2, and v_m does not. Since k_2 is a clique, it follows that v_m has a non-neighbor w in k_2. But now $\{v_1, v_2, v_m, w\}$ induces a diamond, a contradiction. This proves that no induced subgraph of $K(G)$ is an odd hole, and so $K(G)$ is Berge. Finally, Proposition 3.1 completes the proof.

In the case that G is a line graph we prove:

Theorem 3.4 *Let G be a line graph. Then the following are equivalent:*

(i) *G is interesting*

(ii) *no induced subgraph of G is and odd hole, or a 3-generalized sun*

(iii) *G is clique-perfect.*

We prove again that (ii) implies that $K(G)$ is perfect, in order to show that G is clique-perfect. But note that we can not use Proposition 3.1 to prove that $K(G)$ perfect implies (iii), because line graphs are not necessarily HCH.

The last part of Theorem 3.2 is the following:

Theorem 3.5 *Let G be claw-free and assume that G is HCH. Then the following are equivalent:*

(i) *G is interesting*

(ii) *no induced subgraph of G is an odd hole, or $\overline{C_7}$*

(iii) *G is clique-perfect*

In this case, the proof of $K(G)$ perfect is rather involved, and uses a recent structure theorem for claw-free graphs [9], that states that every claw-free graph can be built from a few basic classes by gluing them together in prescribed ways. Thus in order to prove that $K(G)$ is perfect, it is enough to show that it holds for the basic classes, and then that the property is preserved under the gluing operations. However, one of the basic classes we need to deal with is the class of graphs with stability number 2, and part of the proof is getting an explicit description of all HCH graphs in this class. Finally, the use of Proposition 3.1 completes the proof of Theorem 3.5.

These results allow us to formulate partial characterizations of clique-perfect graphs by forbidden subgraphs, as is shown in Table 1.

Graph classes	**Forbidden subgraphs**	**Reference**
Diamond-free graphs	odd generalized suns	Thm 3.3
HCH claw-free graphs	odd holes $\overline{C_7}$	Thm 3.5
Line graphs	odd holes 3-generalized sun	Thm 3.4

Table 1
Forbidden induced subgraphs for clique-perfect graphs in each studied class.

Note that in the last two cases all the forbidden induced subgraphs are minimal. In the first case, however, we need to forbid every odd-generalized sun. Obviously, in this case it is enough to forbid diamond-free odd generalized suns. It is easy to see that all such suns have no improper edges but we do not yet know what the minimal diamond-free odd generalized suns are.

References

[1] V. Balachandhran, P. Nagavamsi and C. Pandu Rangan, *Clique-transversal and clique-independence on comparability graphs*, Inf. Process. Lett. **58** (1996), 181–184.

[2] C. Berge and M. Las Vergnas, *Sur un théorème du type König pour hypergraphes*, Ann. N.Y. Acad. Sci. **175** (1970), 32–40.

[3] F. Bonomo, G. Durán, M. Groshaus and J. Szwarcfiter, *On clique-perfect and K-perfect graphs*, Ars Combinatoria, to appear.

[4] A. Brandstädt, V. Chepoi and F. Dragan, *Clique r-domination and clique r-packing problems on dually chordal graphs*, SIAM J. Discrete Math. **10** (1997), 109–127.

[5] M. Chang, M. Farber and Z. Tuza, *Algorithmic aspects of neighbourhood numbers*, SIAM J. Discrete Math. **6** (1993), 24–29.

[6] L. Chong-Keang and P. Yee-Hock, *On graphs without multicliqual edges*, J. Graph Theory **5** (1981), 443–451.

[7] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour and K. Vušković, *Recognizing Berge Graphs*, Combinatorica, to appear.

[8] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, *The Strong Perfect Graph Theorem*, Annals of Mathematics, to appear.

[9] M. Chudnovsky and P. Seymour, *The Structure of Clawfree Graphs*, Proceedings of the British Combinatorial Conference, Durham (2005), to appear.

[10] G. Durán, M. Lin and J. Szwarcfiter, *On clique-transversal and clique-independent sets*, Ann. Oper. Res. **116** (2002), 71–77.

[11] M. Golumbic, "Algorithmic Graph Theory and Perfect Graphs", Academic Press, New York, 1980.

[12] M. Grötschel, L. Lovász and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica **1** (1981), 169–197.

[13] V. Guruswami and C. Pandu Rangan, *Algorithmic aspects of clique-transversal and clique-independent sets*, Discrete Appl. Math. **100** (2000), 183–202.

[14] J. Lehel and Z. Tuza, *Neighborhood perfect graphs*, Discrete Math. **61** (1986), 93–101.

[15] E. Prisner, *Hereditary clique-Helly graphs*, J. Comb. Math. Comb. Comput. **14** (1993), 216–220.