Characterization and recognition of Helly circular-arc clique-perfect graphs

Flavia Bonomo [a,1,3] and Guillermo Durán [b,2,4]

[a] *Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina.*

[b] *Departamento de Ingeniería Industrial, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile.*

Abstract

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. A graph G is *clique-perfect* if the sizes of a minimum clique-transversal and a maximum clique-independent set are equal for every induced subgraph of G. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem. In this work we characterize clique-perfect graphs by a restricted list of minimal forbidden induced subgraphs when the graph is a Helly circular-arc graph. This characterization leads to a polynomial time recognition algorithm for clique-perfect graphs inside this class of graphs.

Keywords: Clique-perfect graphs, Helly circular-arc graphs, K-perfect graphs, perfect graphs.

1 Introduction

Let G be a simple finite undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$, the complement of G.

A family of sets S is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element. A *circular-arc graph* is the intersection graph of arcs of a circle. A *Helly circular-arc* (HCA) *graph* is a circular-arc graph admitting a model whose arcs satisfy the Helly property.

A *clique* is a complete subgraph maximal under inclusion. A graph is *clique-Helly* (CH) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (HCH) if H is clique-Helly for every induced subgraph H of G.

A graph G is *perfect* when the chromatic number equals the clique number for every induced subgraph of G. It has been proved recently that perfect graphs can be characterized by two families of minimal forbidden induced subgraphs [4] and recognized in polynomial time [3]. The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. A graph G is *K-perfect* if $K(G)$ is perfect.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_c(G)$ and $\alpha_c(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. It is easy to see that $\tau_c(G) \geq \alpha_c(G)$ for any graph G. A graph G is *clique-perfect* if $\tau_c(H) = \alpha_c(H)$ for every induced subgraph H of G. Clique-perfect graphs have been implicitly studied in a lot of works, but the terminology "clique-perfect" has been introduced in [8]. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. Another open question concerning clique-perfect graphs is the complexity of the recognition problem.

There are some partial results in this direction. In [9], clique-perfect graphs are characterized by minimal forbidden subgraphs for the class of chordal

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[3] Email: `fbonomo@dc.uba.ar`

[4] Email: `gduran@dii.uchile.cl`

graphs. In [10], minimal graphs G with $\alpha_c(G) = 1$ and $\tau_c(G) > 1$ are explicitly described. In [1], clique-perfect graphs are characterized by minimal forbidden subgraphs for the classes of line graphs and hereditary clique-Helly claw-free graphs, and by forbidden subgraphs for the class of diamond-free graphs.

In this work, we give a characterization of clique-perfect graphs for the whole class of Helly circular-arc graphs by minimal forbidden subgraphs.

2 Main results

Let G be a graph and C be a cycle of G not necessarily induced. An edge of C is *non proper* if it forms a triangle with some vertex of C. An *r-generalized sun*, $r \geq 3$, is a graph G whose vertex set can be partitioned into two sets: a cycle C of r vertices, with all its non proper edges $\{e_j\}_{j \in J}$ (J is permitted be an empty set) and a stable set $U = \{u_j\}_{j \in J}$, such that for each $j \in J$, u_j is adjacent only to the endpoints of e_j. An r-generalized sun is said to be *odd* if r is odd. Odd generalized suns are not clique-perfect [2], but, unfortunately, they are not necessary minimal (with respect to taking induced subgraphs). However, the odd generalized suns involved in the characterization of HCA clique-perfect graphs by forbidden subgraphs can be described as a union of some families which are minimally clique-imperfect.

A hole is a chordless cycle of length $n \geq 4$, and it is denoted by C_n. A hole C_n is said to be *odd* if n is odd. Clearly odd holes are odd generalized suns.

The graphs S_k^1, S_k^2, S_k^3 and S_k^4 in Figure 1, where $k \geq 2$ and the length of the induced path depicted as a dotted line is $2k - 3$, are minimally clique-imperfect. In particular, S_k^3 and S_k^4 are $2k + 1$-generalized suns.

Theorem 2.1 *Let G be a HCA graph. Then G is clique-perfect if and only if G does not contain any of the graphs of Figure 1 as an induced subgraph.*

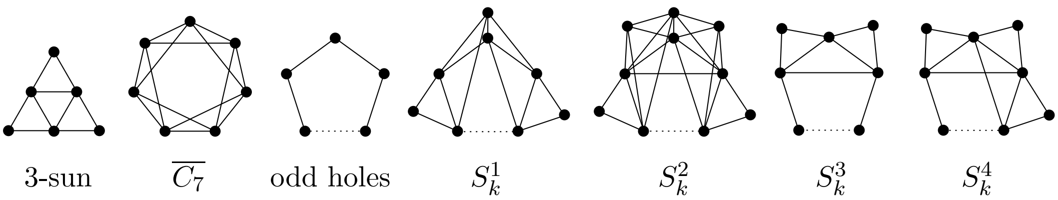

Fig. 1. Minimal forbidden subgraphs for clique-perfect graphs inside the class of HCA graphs. Dotted lines replace any induced path of odd length at least 1.

Moreover, we prove that a HCA graph which does not contain any of the graphs of Figure 1 as an induced subgraph is K-perfect. In general, clique-perfect graphs are not necessarily K-perfect, and conversely. But, if a hereditary graph class is HCH and K-perfect, then it is clique-perfect. We use that in the proof of Theorem 2.1, and handle separately the case of $HCA \setminus HCH$.

Helly circular-arc graphs can be recognized in polynomial time [7] and, given a Helly model of a HCA graph G, both parameters $\tau_c(G)$ and $\alpha_c(G)$ can be computed in linear time [5,6]. However, clearly it is not straightforward from these properties the existence of a polynomial time recognition algorithm for clique-perfect HCA graphs. The characterization in Theorem 2.1 leads to such an algorithm, which is strongly based on the recognition of perfect graphs [3].

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