Between coloring and list-coloring: μ-coloring

Flavia Bonomo[1,2] and Mariano Cecowski[3]

Departamento de Computación
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Buenos Aires, Argentina

Abstract

A new variation of the coloring problem, μ-coloring, is defined in this paper. Given a graph G and a function μ, a μ-coloring is a coloring where each vertex v of G must receive a color at most $\mu(v)$. It is proved that μ-coloring lies between coloring and list-coloring, in the sense of generalization of problems and computational complexity. The notion of perfection is extended for μ-coloring, leading us to a new characterization of cographs. Finally, a polynomial time algorithm to solve μ-coloring for cographs is shown.

Keywords: cographs, coloring, list-coloring, μ-coloring, M-perfect graphs, perfect graphs.

---

[1] Partially supported by UBACyT Grant X184, PICT ANPCyT Grant 11-09112 and PID Conicet Grant 644/98, Argentina CNPq under PROSUL project Proc. 490333/2004-4, Brazil.
[2] Email: `fbonomo@dc.uba.ar`
[3] Email: `mcecowsk@dc.uba.ar`

1 Introduction

Let G be a graph, with vertex set $V(G)$. Denote by $N_G(v)$ the set of neighbors of $v \in V(G)$. A cograph is a P_4-free graph, where P_4 is the path of four vertices.

A *complete* of G is a subset of vertices pairwise adjacent. A *clique* is a complete not properly contained in any other. We may also use the term clique to refer to the corresponding complete subgraph. Let X and Y be two sets of vertices of G. We say that X is *complete to* Y if every vertex in X is adjacent to every vertex in Y, and that X is *anticomplete to* Y if no vertex of X is adjacent to a vertex of Y.

A coloring of a graph $G = (V, E)$ is a function $f : V \to \mathbb{N}$ such that $f(v) \neq f(w)$ if v is adjacent to w. A k-coloring is a coloring f for which $f(v) \leq k$ for every $v \in V$. A graph G is k-colorable if there is a k-coloring of G.

Several variations of the coloring problem are studied in the literature (see a review in [10], and a recent work in [8]). One of them is list-coloring [11]. Given a graph $G = (V, E)$ and a finite list $L(v) \subseteq \mathbb{N}$ of colors for each vertex $v \in V$, G is list-colorable if there is a coloring f for which $f(v) \in L(v)$ for each $v \in V$.

We define here μ-coloring as follows. Given a graph $G = (V, E)$ and a function $\mu : V \to \mathbb{N}$, G is μ-colorable if there is a coloring f for which $f(v) \leq \mu(v)$ for each $v \in V$. This problem lies between k-coloring and list-coloring. A trivial reduction from k-coloring to μ-coloring can be done defining $\mu(v) = k$ for every v. The reduction from μ-coloring to list-coloring can be done defining $L(v) = \{1, \ldots, \min\{\mu(v), |V(G)|\}\}$. We show in this work that the betweenness is strict, that is, there is a class of graphs (bipartite graphs) for which μ-coloring is NP-complete while coloring is in P, and there is another class of graphs (cographs) for which list-coloring is NP-complete while μ-coloring is in P.

We say that a coloring f is *minimal* when for every vertex v, and every $i < f(v)$, v has a neighbor w_i with $f(w_i) = i$. Note that every k-coloring or μ-coloring can be transformed into a minimal one.

The chromatic number of a graph G is the minimum k such that G is k-colorable, and is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of G, the clique number of G, denoted by $\omega(G)$. A graph G is perfect [1] when $\chi(H) = \omega(H)$ for every induced subgraph H of G. Perfect graphs have very nice properties: they are a self-complementary class of graphs [9], the k-coloring problem is solvable in polynomial time for perfect graphs [5], they have been characterized by minimal forbidden subgraphs [2] and recognized in polynomial time [3].

In this work we define M-perfect graphs and show that they are exactly the cographs. It follows from this equivalence that M-perfect graphs are a self-complementary class of graphs and can be recognized in linear time [4]. Moreover, we show that the μ-coloring problem is solvable in polynomial time for this class of graphs.

2 Cographs and M-perfect graphs

A graph G is perfect when $\chi(H) = \omega(H)$ for every induced subgraph H of G. This definition is equivalent to the following: "G is perfect when for every induced subgraph H of G and for every k, H is k-colorable if and only if every clique of H is k-colorable".

Analogously, we define M-perfect graphs as follows: a graph G is M-perfect when for every induced subgraph H of G and for every function $\mu : V \to \mathbb{N}$, H is μ-colorable if and only if every clique of H is μ-colorable.

M-perfect graphs are also perfect, because perfection is equivalent to M-perfection with μ restricted to constant functions. The converse is not true. We will show that the graph P_4 is not M-perfect, although it is perfect. In fact, M-perfect graphs are exactly the cographs. In order to prove it we need the next general result about minimal colorings on cographs.

Theorem 2.1 *Let G be a cograph and $x \in V(G)$. Let f be a minimal coloring of $G - x$, and $T \in \mathbb{N}$. If f cannot be extended to G coloring x with a color at most T then there is a complete $H \subseteq N_G(x)$ of size T and such that $f(H) = \{1, \ldots, T\}$.*

Proof. Let G be a cograph and $x \in V(G)$. Let f be a minimal coloring of $G - x$, and $T \in \mathbb{N}$. Let us prove the result by induction on T. Suppose first that $T = 1$. If f cannot be extended to G coloring x with color 1, then there exists $v \in N_G(x)$ such that $f(v) = 1$. In this case, $H = \{v\}$. Suppose it holds for $T = s - 1$ and let us see that it holds for $T = s$, $s \geq 2$. If f cannot be extended to G coloring x with a color less or equal to s, in particular the same holds for $s-1$, and so, by inductive hypotheses, there is a complete $H \subseteq N_G(x)$ of size $s-1$ using the colors from 1 to $s-1$. On the other hand, since x cannot use color s, it must be a vertex $v \in N_G(x)$ such that $f(v) = s$. Let us consider the subgraph $\widetilde{G}$ of $G-x$ induced by $\{w \in G-x : f(w) \leq s-1\} \cup \{v\}$ and let $\widetilde{f}$ be the coloring f restricted to $\widetilde{G} - v$. By the minimality of f it follows that $\widetilde{f}$ is minimal and it cannot be extended to $\widetilde{G}$ coloring v with a color less or equal than $s - 1$, so, by inductive hypotheses, there is a complete $F \subseteq N_{\widetilde{G}}(v)$

of size $s-1$ using colors from 1 to $s-1$.
If $H = F$ then $H \cup \{v\}$ is a complete of size s in the neighborhood of x using colors from 1 to s. Suppose that they are not equal. Then $F \setminus H$ and $H \setminus F$ have the same cardinality and use the same colors. Let v_H in $H \setminus F$, and let v_F in $F \setminus H$ such that $f(v_F) = f(v_H)$. Since f is a coloring of $G - x$, v_F and v_H are not adjacent. Since G is P_4-free, v_H, x, v, v_F do not induce a P_4, so x is adjacent to v_F or v is adjacent to v_H. If all the vertices of $H \setminus F$ are adjacent to v, then $H \cup \{v\}$ is a complete of size s in the neighborhood of x using colors from 1 to s.
So, suppose that the set $H_v = \{w \in H : (w, v) \notin E(G)\}$ is non empty, and define $F_v = \{z \in F : \exists z_H \in H_v \text{ with } f(z) = f(z_H)\}$. Note that F_v and H_v have the same cardinality and use the same colors. Since H_v is anticomplete to v, it follows that F_v must be complete to x. If $H \setminus H_v$ is empty, then $F = F_v$ is complete to x and $F \cup \{v\}$ is a complete of size s in the neighborhood of x using colors from 1 to s.
Suppose that $H \setminus H_v$ is non empty, and let us see that F_v is complete to $H \setminus H_v$. Let $z \in F_v$ and $w \in H \setminus H_v$. Let $z_H \in H_v$ such that $f(z_H) = f(z)$. Then z_H is neither adjacent to z nor to v and since H is a complete, z_H and w are adjacent. Besides, w is adjacent to v because of being in $H \setminus H_v$. Since z_H, w, v, z do not induce a P_4, w must be adjacent to z. Therefore F_v is complete to $H \setminus H_v$. Hence $\widetilde{H} = (H \cup F_v \cup \{v\}) \setminus H_v$ is a complete in $N_G(x)$ of size s such that $f(\widetilde{H}) = \{1, \dots, s\}$. □

Theorem 2.2 *If G is a graph, the following are equivalent:*

(i) *G is a cograph*

(ii) *G is M-perfect*

(iii) *for every function $\mu : V \to \mathbb{N}$, G is μ-colorable if and only if every clique of G is μ-colorable.*

Proof (Sketch). It is easy to prove that (ii) and (iii) are equivalent. Let us see that (i) and (ii) are equivalent.

(ii) $\Rightarrow$ (i)) Let $v_1 v_2 v_3 v_4$ be a P_4, and let μ be defined as follows: $\mu(v_1) = \mu(v_4) = 1$, $\mu(v_2) = \mu(v_3) = 2$. Clearly, every clique is μ-colorable, but the whole graph is not.

(i) $\Rightarrow$ (ii)) Suppose that there is a P_4-free graph which is not M-perfect. Let G be a minimal one, that is, G is P_4-free and it is not M-perfect, but for every vertex x of G, $G - x$ is M-perfect.
Let $\mu : V(G) \to \mathbb{N}$ such that the cliques of G are μ-colorable but G is not. Let x be a vertex of G with $\mu(x)$ maximum. The graph $G - x$ is M-perfect,

and since the cliques of G are μ-colorable, also those of $G-x$ are, so $G-x$ is μ-colorable. Let f be a minimal μ-coloring of $G-x$.
Since G is not μ-colorable, f cannot be extended to a μ-coloring of G. Hence by Theorem 2.1, $N_G(x)$ contains a complete of size $\mu(x)$. But then G contains a complete of size $\mu(x)+1$ for which the upper bounds of all of its vertices are at most $\mu(x)$ (we have chosen x with maximum value of μ). This is a contradiction, because all the cliques of G are μ-colorable.
Therefore there is not minimal M-imperfect P_4-free graph, and that concludes the proof. □

3 Algorithm for μ-coloring cographs

The greedy coloring algorithm consists of successively color the vertices with the least possible color in a given order.

From Theorem 2.1 we can prove the following result.

Theorem 3.1 *The greedy coloring algorithm applied to the vertices in non-decreasing order of μ gives a μ-coloring for a cograph, when it is μ-colorable.*

A little improvement in the greedy algorithm allows us to find a non μ-colorable clique when the graph is not μ-colorable. A nice corollary of this theorem is the following.

Corollary 3.2 *The greedy coloring algorithm gives an optimal coloring for cographs, independently of the order of the vertices.*

Jansen and Scheffler [7] prove that list-coloring is NP-complete for cographs, hence μ-coloring is “easier” than list-coloring, unless P=NP.

4 Bipartite graphs

It follows from Theorem 2.1 that a cograph G that is μ-colorable can be μ-colored using the first $\chi(G)$ colors. This does not happen for bipartite graphs, even for trees.

Define the family $\{T_n\}_{n\in\mathbb{N}}$ of trees and the corresponding family $\{\mu_n\}_{n\in\mathbb{N}}$ of functions as follows: $T_1=\{v\}$ is a trivial tree, and $\mu_1(v)=1$. The tree T_{n+1} is obtained from $T_1,\dots,T_n$ by adding a root w adjacent to the roots of $T_1,\dots,T_n$. Function μ_{n+1} extends $\mu_1,\dots,\mu_n$ and is defined at w as $\mu_{n+1}(w)=n+1$. The tree T_n requires n colors to be μ_n-colored, and it has 2^{n-1} vertices. In fact, the following property holds.

Theorem 4.1 *Let T be a tree, and let μ be a function such that T is μ-colorable. Then T can be μ-colored using at most the first $log_2(|V(T)|) + 1$ colors.*

A similar result can be obtained for bipartite graphs. Define the family $\{B_n\}_{n\in\mathbb{N}}$ of bipartite graphs and the corresponding family $\{\mu_n\}_{n\in\mathbb{N}}$ of functions as follows: $B_1 = \{v\}$ is a trivial graph, and $\mu_1(v) = 1$. The bipartite graph $B_{n+1} = (V, W, E)$ has $V = \{v_1, \ldots, v_n\}$, $W = \{w_1, \ldots, w_n\}$; v_i is adjacent to w_j for $i \neq j$; v_n is adjacent to w_n, and v_i is not adjacent to w_i for $i < n$; $\mu_{n+1}(v_i) = \mu_{n+1}(w_i) = i$ for $i < n$; $\mu_{n+1}(v_n) = n$ and $\mu_{n+1}(w_n) = n + 1$. The bipartite graph B_n requires n colors to be μ_n-colored, and it has $2n - 2$ vertices (if $n \geq 2$). Analogously, the following property holds.

Theorem 4.2 *Let B be a bipartite graph, and let μ be a function such that B is μ-colorable. Then B can be μ-colored using at most the first $\frac{(|V(B)|+2)}{2}$ colors.*

Hujter and Tuza [6] prove that list-coloring is NP-complete for bipartite graphs, and the same holds for μ-coloring.

Theorem 4.3 *μ-coloring is NP-complete for bipartite graphs.*

Proof. Consider an instance of bipartite list-coloring, i.e., assume that a bipartite graph $G = (X, Y, E)$ is given, and for each $v \in V(G)$, we have a finite list $L(v) \subseteq \mathbb{N}$ of the possible colors of v. Let $k = |\bigcup_{v\in V(G)} L(v)|$. Without loss of generality we can assume that $L(v) \subseteq \{1, \ldots, k\}$. We add two k-element sets of vertices, $X' = \{x'_1, \ldots, x'_k\}$ and $Y' = \{y'_1, \ldots, y'_k\}$ to G such that X, Y, X', Y' are pairwise disjoint. Furthermore, we take a bipartition $(X \cup X', Y \cup Y')$ of the new graph G', and for any $x \in X$, $y \in Y$, and $i, j \in \{1, \ldots, k\}$, define the following new adjacency relations: x'_i is adjacent to y'_j if and only if $i \neq j$; x'_i is adjacent to y if and only if $i \notin L(y)$; y'_i is adjacent to x if and only if $i \notin L(x)$. We define $\mu(x'_i) = \mu(y'_i) = i$ and $\mu(x) = \mu(y) = k$. Then G is list-colorable if and only if G' is μ-colorable. The transformation can be made in polynomial time, and this completes the proof. □

Coloring is trivially in P for bipartite graphs, hence μ-coloring is "harder" than coloring, unless P=NP.

Acknowledgement

We thank Guillermo Durán and Adrian Bondy for their valuable ideas which improved this work.

References

[1] C. Berge, *Les problemes de colorations en théorie des graphes*, Publ. Inst. Stat. Univ. Paris **9** (1960), 123–160.

[2] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, *The Strong Perfect Graph Theorem*, Annals of Mathematics, to appear.

[3] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour and K. Vušković, *Recognizing Berge Graphs*, Combinatorica, to appear.

[4] D. Corneil, Y. Perl and L. Stewart, *Cographs: recognition, applications and algorithms*, Congr. Numer. **43** (1984), 249–258.

[5] M. Grötschel, L. Lovász and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica **1** (1981), 169–197.

[6] M. Hujter and Zs. Tuza, *Precoloring extension. II. Graph classes related to bipartite graphs*, Acta Math. Univ. Comenianae **62** (1) (1993), 1–11.

[7] K. Jansen and P. Scheffler, *Generalized coloring for tree-like graphs*, Discrete Applied Mathematics **75** (1997), 135–155.

[8] S. Klein and M. Kouider, *On b-perfect graphs*, In: Annals of the XII CLAIO, Havanna, Cuba (2004).

[9] L. Lovász, *A characterization of perfect graphs and the perfect graph conjecture*, J. Combin. Theory B **132** (1972), 95–98.

[10] Zs. Tuza, *Graph colorings with local constraints – a survey*, Discuss. Math. Graph Theory **17** (1997), 161–228.

[11] V. Vizing, *Coloring the vertices of a graph in prescribed colors*, Metody Diskret. Anal. v Teorii Kodov i Shem **29** (1976), 3–10.