# Self-clique Helly circular-arc graphs

Flavia Bonomo [1]

*Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina.* `fbonomo@dm.uba.ar`

**Abstract**

A *clique* in a graph is a complete subgraph maximal under inclusion. The *clique graph* of a graph is the intersection graph of its cliques. A graph is *self-clique* when it is isomorphic to its clique graph. A *circular-arc graph* is the intersection graph of a family of arcs of a circle. A *Helly circular-arc graph* is a circular-arc graph admitting a model whose arcs satisfy the Helly property. In this note, we describe all the self-clique Helly circular-arc graphs.

*Key words:* Helly circular-arc graphs, self-clique graphs.

## 1 Introduction

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.

A *clique* in a graph is a complete subgraph maximal under inclusion. The *clique graph* $K(G)$ of $G$ is the intersection graph of the cliques of $G$. The $j$-th iterated clique graph of $G$, $K^j(G)$, is defined by $K^1(G) = K(G)$ and $K^j(G) = K(K^{j-1}(G))$, $j \geq 2$.

A graph $G$ is *self-clique* when $K(G) \cong G$, i.e., $G$ is isomorphic to its clique graph. More generally, for $t \geq 1$, a graph $G$ is *$t$-self-clique* if $K^t(G) \cong G$ and $K^j(G) \ncong G$ for $1 \leq j < t$. A graph $G$ is *clique-convergent* if $K^t(G)$ is the one-vertex graph for some $t \geq 1$.

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A *circular-arc graph* is the intersection graph of a family of arcs of a circle. (Without loss of generality, we can assume that the arcs are open.) Basic background in circular-arc graphs can be found in [9]. A family of sets $S$ is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element. A *Helly circular-arc* ($HCA$) *graph* is a circular-arc graph admitting a model whose arcs satisfy the Helly property. A circular-arc model of a graph is *proper* if no arc is included in another. A *proper circular-arc* ($PCA$) *graph* is a circular-arc graph admitting a proper model. A graph is *clique-Helly* ($CH$) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* ($HCH$) if $H$ is clique-Helly for every induced subgraph $H$ of $G$.

Clique graphs of Helly circular-arc graphs are characterized in [7]. It is proved that they are a proper subclass of $PCA \cap HCA \cap CH$.

A graph is *chordal* when every cycle of length at least four has a chord. A common subclass of chordal graphs and circular-arc graphs are interval graphs. An *interval graph* is the intersection graph of a family of intervals in the real line.

Self-clique graphs were studied in [1,2,4,6,8,11–13], but no good general characterization of them is known. However, self-clique and 2-self-clique graphs are characterized for some classes of graphs, like triangle-free graphs [8], graphs with all cliques but one of size 2 [6], clique-Helly graphs [4,8,12] and hereditary clique-Helly graphs [13].

For $v \in V(G)$, denote by $N(v)$ the set of neighbors of $v$. Let $N[v] = \{v\} \cup N(v)$. The vertex $v$ is *dominated* by vertex $w$ if $N[v] \subseteq N[w]$. In [8] it is proved that a clique-Helly graph $G$ is $t$-self-clique (for some $t$) if and only if it has no dominated vertices, and in that case $t \leq 2$.

For some classes of graphs, it can be proved that there are no self-clique graphs. For example, in [3,5] it is proved that every connected chordal graph is clique-convergent. So there are no chordal $t$-self-clique graphs with at least one edge.

In this note, we give an explicit characterization of self-clique graphs for the class of Helly circular-arc graphs.

## 2 Characterization

Given a graph $G$ and $k \geq 0$, the graph $G^k$ has the same vertex set of $G$, two vertices being adjacent in $G^k$ if their distance in $G$ is at most $k$. Denote by

$C_n$ the chordless cycle on $n$ vertices.

Graphs $C_n^k$, with $n > 3k$, are Helly circular-arc graphs (some examples can be seen in Figure 1). Besides, in [10] it is proved that graphs $C_n^k$, with $n > 3k$, are self-clique graphs.

**Theorem 1** *Let $G$ be a HCA graph with $n$ vertices. Then the following are equivalent:*

*(i) $G$ is $t$-self-clique for some $t \geq 1$*
*(ii) $G$ is self-clique*
*(iii) $G$ is isomorphic to $C_n^k$ for some $k \geq 0$ such that $3k < n$.*

**PROOF.** *(iii)* $\Rightarrow$ *(ii)*. It is proved in [10].

*(ii)* $\Rightarrow$ *(i)*. It is clear.

*(i)* $\Rightarrow$ *(iii)*. Let $G$ be a $HCA$ graph with $n$ vertices. If $G$ has no edges, then it is isomorphic to $C_n^0$. So, suppose that $G$ is $t$-self-clique for some $t \geq 1$ and it has at least one edge. Then every circular-arc model of $G$ covers the circle, otherwise $G$ would be an interval graph, and there are no chordal $t$-self-clique graphs with at least one edge.

The graph $K(G)$ is clique-Helly [7], and since clique-Helly is a fixed class under the clique operator $K$ [8,3], then $G \cong K^t(G)$ is clique-Helly and then it is either self-clique or 2-self-clique and it has no dominated vertices [8]. As a consequence of this, every circular-arc model of $G$ is proper, and, in particular, $G$ has a circular-arc model which is both Helly and proper.

In a Helly circular-arc model of $G$, for every clique there is a point of the circle that belongs to the arcs corresponding to the vertices in the clique, and to no others. We call such a point an *anchor* of the clique (please note that an anchor may not be unique). If there are two arcs covering the circle, their corresponding vertices are adjacent and belong to a clique $M$. Every other clique contains at least one of those vertices, so $M$ intersects all the cliques of $G$, and then $K^2(G)$ is complete and $G$ is clique-convergent, so $G$ cannot be $t$-self clique because it contains at least one edge. Therefore no two arcs cover the circle, and, as it is a Helly model, no three arcs cover the circle.

Traversing an arc $A_i$ clockwise, its endpoints can be identified as a head $a_i$ and a tail $b_i$. Without loss of generality (see Exercise 8.14 in [9]), we can consider that the endpoints of the arcs are $2n$ distinct points of the circle, and we can choose the anchors for the distinct cliques of $G$ in the interior of the $2n$ circular intervals determined by those $2n$ points. In each of these intervals there are anchors of at most one clique, and, in fact, only the intervals of type

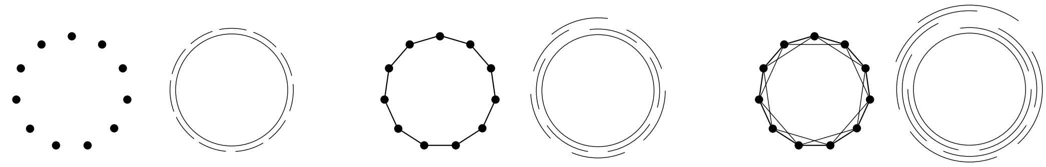

Fig. 1. From left to right, graphs $C_{11}^0, C_{11}^1$ and $C_{11}^2$, with their corresponding Helly circular-arc model.

$a_i, b_j$ (clockwise) can contain anchors. So $G$ has $r \leq n$ cliques, and, as this argument can be applied to $K(G)$ because it is a $HCA$ graph [7], $K^2(G)$ has at most $r$ vertices, so $r = n$. Therefore, heads and tails are alternating, and since the model is proper the clockwise order of the heads must be the same as the clockwise order of the tails. Thus $G$ is uniquely determined by the number $k$ of heads in the interior of the arc $A_1$, and therefore $G$ is isomorphic to $C_n^k$. Finally, since no three of the arcs cover the circle, it follows that $3k < n$. $\square$

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