Cleaning for Bergeness

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Abstract

A graph is *Berge* if no induced subgraph of G is an odd cycle of length at least 5 or the complement of one. Let C be a shortest odd hole in G. A vertex v is *major* for C if its neighbours in C do not lie in a 2-edge path of C. In this paper we give a polynomial time algorithm, which with input a graph G, either decides that G is not Berge, or outputs polynomially many subsets of $V(G)$, with the following property: if C is a shortest odd hole in G, and every major vertex has at least four neighbours in C, then one of the subsets is disjoint from C and contains every vertex that is major for C.

This is half of an algorithm to test if a graph is Berge. (The other half is done in two different ways, in two companion papers in this issue.)

1 Introduction

The objective of this paper is to describe an algorithm that is applied in the two companion papers in this issue [2, 3]. We refer the reader to those papers for motivation, but we repeat the definitions that we require. All graphs in this paper are finite, and simple. The vertex- and edge-sets of a graph G are denoted by $V(G), E(G)$. The *complement* of G is the graph $\overline{G}$, with vertex set $V(G)$, in which two distinct vertices are adjacent if and only if they are nonadjacent in G. A subset $X \subseteq V(G)$ is *connected* if the subgraph of G induced on X is connected; and *anticonnected* if the subgraph of $\overline{G}$ induced on X is connected. If $X \subseteq V(G)$, a *component* of X means a maximal nonnull connected subset of X, and an *anticomponent* of X is defined similarly. A vertex v is *X-complete* if $v \in V(G) \setminus X$ is adjacent to every vertex in X. A *hole* in G is an induced cycle in G of length ≥ 4, and an *antihole* is an induced subgraph of G whose complement is a hole. A *path* in G is an induced subgraph that is connected, with at least one vertex, no cycle, and no vertex of degree > 2. The *ends* of a path are defined as usual. We denote the set of internal vertices of a path P (that is, its *interior*) by P^*. An *antipath* is an induced subgraph whose complement is a path. The *length* of a path or hole is the number of edges in it, and the *length* of an antipath or antihole is the length of its complement. G is *Berge* if there is no hole or antihole in G of odd length.

Let C be a shortest odd hole in a graph G. A vertex $v \in V(G)$ is *major* for C if its neighbours in C do not lie in a 2-edge path of C; and a major vertex is a *racket* if it has exactly three neighbours in C. Let us say a *cleaner* for C is a subset $X \subseteq V(G)$, containing every vertex that is major for C, and such that $X \cap V(C)$ belongs to a 2-edge path of C. The object of this paper is to prove the following.

1.1 *There is an algorithm with the following specifications:*

- **Input** *A graph G.*
- **Output** *Either*
  1. *$O(|V(G)|^5)$ subsets, such that if C is a shortest odd hole in G with no racket, then one of the subsets is a cleaner for C, or*
  2. *it determines that G is not Berge.*
- **Running time** *$(|V(G)|^6)$.*

The proof of 1.1 will occupy the remainder of the paper. The main idea is to examine the common neighbours of anticonnected sets of major vertices, in a shortest odd hole.

2 Some easily-detectable subgraphs

Before we look at general anticonnected sets of major vertices, we need to exclude some easily-detectable non-Berge subgraphs.

1. A configuration of type $\mathcal{T}_1$ in G is a hole of length 5.
2. A configuration of type $\mathcal{T}_2$ in G is a sequence $v_1, \ldots, v_5, Q$ such that $v_1, \ldots, v_5$ are distinct vertices, v_1-v_2-v_3-v_4 is a path, v_5 is nonadjacent to v_2 and v_3, and Q is an antipath of G between v_2, v_3 such that all internal vertices of Q are $\{v_1, v_4, v_5\}$-complete.

3. A configuration of type $\mathcal{T}_3$ in G is a sequence v_1, v_2, v_3, v_4, P, X such that:
   - v_1-v_2-v_3-v_4 is a path of G
   - X is an anticomponent of the set of all $\{v_1, v_2, v_4\}$-complete vertices
   - P is a path in $G \setminus v_2, v_3$ between v_1, v_4, and no vertex in P^* is X-complete or adjacent to v_2 or v_3.
4. A configuration of type $\mathcal{T}_4$ in G is a sequence $v_1, \ldots, v_6, P, X$ such that
   - $v_1, \ldots, v_6$ are distinct vertices of G
   - $v_1v_2, v_3v_4, v_1v_4, v_2v_3, v_3v_5, v_4v_6$ are edges, and $v_1v_3, v_2v_4, v_1v_5, v_2v_5, v_1v_6, v_2v_6, v_4v_5$ are non-edges
   - X is an anticomponent of the set of all $\{v_1, v_2, v_5\}$-complete vertices, and v_3, v_4 have nonneighbours in X
   - P is a path of $G \setminus (X \cup \{v_1, v_2, v_3, v_4\})$ between v_5, v_6, and no internal vertex of P is X-complete, and no internal vertex of P is adjacent to v_1 or to v_2
   - if v_5v_6 is an edge then v_6 is not X-complete.

Clearly we can test whether G contains a configuration of type $\mathcal{T}_1$ in time $O(|V(G)|^5)$, and if so then G is not Berge. A configuration of type $\mathcal{T}_2$ is just the complement of what is called a *jewel* in [2] (with the terms of the sequence in a different order). The results of that paper therefore imply that there is an algorithm which will test whether G contains a configuration of type $\mathcal{T}_2$ with running time $O(|V(G)|^6)$; and if so then G is not Berge. We need analogous results for the other two types of configurations.

2.1 *There is an algorithm with the following specifications:*

- **Input** *A graph G.*
- **Output** *Decides whether G contains a configuration of type $\mathcal{T}_3$.*
- **Running time** $O(|V(G)|^6)$.

Proof. Here is an algorithm. Enumerate all paths v_1-v_2-v_3-v_4 of G. For each one, find the set Y of all $\{v_1, v_2, v_4\}$-complete vertices. Find the anticomponents of Y. For each anticomponent X, test if there is a path P between v_1, v_4 in $G \setminus \{v_2, v_3\}$, such that no internal vertex of P is adjacent to v_2 or v_3, and no internal vertex of P is X-complete. This evidently performs as claimed. This proves 2.1. ∎

The following is proved in [1].

2.2 *Let G be Berge, let X be an anticonnected subset of $V(G)$, and P be a path in $G \setminus X$ with odd length, such that both ends of P are X-complete, and no edge of P is X-complete. Then every X-complete vertex has a neighbour in P^*.*

We deduce:

2.3 *If G contains a configuration of type $\mathcal{T}_3$ then G is not Berge.*

Proof. Let v_1, v_2, v_3, v_4, P, X be a configuration of type $\mathcal{T}_3$ in G. If P is even then v_1-v_2-v_3-v_4-P-v_1 is an odd hole, while if P is odd then since the X-complete vertex v_2 has no neighbour in the interior of P, and the ends of P are X-complete and the internal vertices are not, it follows from 2.2 that G is not Berge. This proves 2.3. ∎

Now let us do the same for configurations of type $\mathcal{T}_4$.

2.4 *There is an algorithm with the following specifications:*

- **Input** *A graph G.*
- **Output** *Decides whether G contains a configuration of type $\mathcal{T}_4$.*
- **Running time** $O(|V(G)|^6)$.

Proof. Here is an algorithm. Enumerate all 3-tuples v_1, v_2, v_5 of distinct vertices such that v_1v_2 is an edge and v_5 is nonadjacent to both v_1, v_2. For each choice of v_1, v_2, v_5, find the set Y of all $\{v_1, v_2, v_5\}$-complete vertices, and find its anticomponents. For each anticomponent X, find the maximal connected subset F' containing v_5 with the properties that v_1, v_2 have no neighbours in F' and no vertex of F' is X-complete. Let F be the union of F' and the set of all X-complete vertices that are nonadjacent to all of v_1, v_2, v_5 and have a neighbour in F'. Still with the same choice of v_1, v_2, v_5 and X, we enumerate all vertices v_4 that are adjacent to v_1 and not to v_2, v_5, with a neighbour in F and a nonneighbour in X. For each choice of v_4, we test whether there is a vertex v_3, adjacent to v_2, v_4, v_5 and not to v_1, with a nonneighbour in X. If we find such a vertex v_3, let v_6 be a neighbour of v_4 in F, and P a path from v_6 to v_5 with interior in F'; then $v_1, \ldots, v_6, P, X$ is a configuration of type $\mathcal{T}_4$; output that fact. If after checking all choices of v_1, v_2, v_5, X, v_4 we find no such v_3, then there is no such configuration; output that.

This algorithm evidently tests correctly for configurations of type $\mathcal{T}_4$. To see its running time, there are $O(|V(G)|^3)$ 3-tuples v_1, v_2, v_5 to examine. For each, there are linearly many choices of X, and each one take time $O(|V(G)|^2)$ to process. Then there are linearly many choices of v_4, and each takes linear time to process. So the total running time is $O(|V(G)|^6)$. This proves 2.4. ∎

Next we need to show that no Berge graph contains a configuration of type $\mathcal{T}_4$. We need the following, the Roussel-Rubio lemma [4].

2.5 *Let G be Berge, let X be an anticonnected subset of $V(G)$, and P be a path p_1-$\cdots$-p_n in $G \setminus X$ with odd length ≥ 5, such that p_1, p_n are X-complete and $p_2, \ldots, p_{n-1}$ are not. Then there exist nonadjacent $x, y \in X$ so that there are exactly two edges between x, y and P^*, namely xp_2 and yp_{n-1}.*

We also need:

2.6 *Let G be Berge, and let $X \subseteq V(G)$ be connected. Let $v_1, v_2, v_3, v_4, p_1, p_2$ be distinct vertices of $G \setminus X$, such that*

- *v_1v_3 and v_2v_4 are edges, and there are no edges between $\{v_1, v_3\}$ and $\{v_2, v_4\}$*

- *v_3, v_4 have neighbours in X, and v_1, v_2 do not*
- *p_1, p_2 are nonadjacent; p_1 is adjacent to v_1, v_2, v_3 and not to v_4, and p_2 is adjacent to v_1, v_2, v_4 and not to v_3*
- *there is a path between v_3, v_4 with interior in X such that p_1, p_2 have no neighbours in its interior*
- *no vertex of X is adjacent to both v_3, v_4.*

Then p_1, p_2 have no neighbours in X.

Proof. Suppose that one of p_1, p_2 has a neighbour in X. Let Q be a path between v_3, v_4 with interior in X such that p_1, p_2 have no neighbours in its interior. Let $x_1\text{-}\cdots\text{-}x_k$ be a path of vertices of X such that one of p_1, p_2 is adjacent to x_1, and x_k has a neighbour in Q^*, with k minimum. (This exists because Q^* is nonempty and X is connected.) It follows that none of $x_1, \ldots, x_{k-1}$ have neighbours in Q^*, and none of $x_2, \ldots, x_k$ are adjacent to p_1 or p_2. Let A, B be paths from x_1 to v_3, v_4 respectively with interior in $\{x_2, \ldots, x_k\} \cup Q^*$. From the symmetry we may assume that x_1 is adjacent to p_1. Then B can be completed to a hole via $v_4\text{-}v_2\text{-}p_1\text{-}x_1$, and therefore B has odd length. Consequently it cannot be completed to a hole via $v_4\text{-}p_2\text{-}v_1\text{-}p_1\text{-}x_1$, and so p_2 has neighbours in $B \setminus \{v_4\}$. Since p_2 has no neighbours in $Q^* \cup \{x_2, \ldots, x_k\}$ it follows that p_2 is adjacent to x_1; and this restores the symmetry between p_1 and p_2. Since B is odd, it cannot be completed to a hole via $v_4\text{-}p_2\text{-}x_1$, and so x_1 is adjacent to v_4. Similarly x_1 is adjacent to v_3, contrary to the last hypothesis of the theorem. This proves 2.6. ∎

These two lemmas are applied to prove the following.

2.7 *If G contains a configuration of type $\mathcal{T}_4$ then G is not Berge.*

Proof. Let G be Berge, and suppose that $v_1, \ldots, v_6, P, X$ is a configuration of type $\mathcal{T}_4$ in G. Let Q be an antipath between v_3, v_4 with interior in X. It can be completed to an antihole via $v_4\text{-}v_5\text{-}v_1\text{-}v_3$, and therefore Q is odd. Hence $v_1\text{-}v_3\text{-}Q\text{-}v_4\text{-}v_2$ is an odd antipath. By 2.5, applied in $\overline{G}$ to the path $v_1\text{-}v_3\text{-}Q\text{-}v_4\text{-}v_2$ and anticonnected set $V(P)$, we deduce that there exist $p_1, p_2 \in V(P)$, adjacent in G, such that the only nonedges between p_1, p_2 and $V(Q)$ are p_1v_4, p_2v_3. By 2.6 applied in $\overline{G}$, we deduce that either some $z \in X$ is nonadjacent to both v_3, v_4, or p_1, p_2 are both X-complete. The first is impossible since $z\text{-}v_1\text{-}v_4\text{-}v_3\text{-}v_5\text{-}z$ is not an odd hole. The second is impossible since no internal vertex of P is X-complete, and if v_5, v_6 are adjacent then by hypothesis v_6 is not X-complete. So G contains no such configuration. This proves 2.7. ∎

In summary, then, we have

2.8 *If G or $\overline{G}$ contains a configuration of type $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ or $\mathcal{T}_4$, then G is not Berge. Moreover, there is an algorithm to test whether G or $\overline{G}$ contains such a configuration, with running time $O(|V(G)|^6)$.*

3 Normal subsets

Let C be a cycle, and let $A \subseteq V(C)$. An *A-gap* is a subgraph of C composed of a component X of $C \setminus A$, the vertices of A with neighbours in X, and the edges between A and X. (So if some two vertices in A are nonadjacent, an A-gap is a path in C of length ≥ 2, with both ends in A and no internal vertex in A.) The *length* of an A-gap is the number of edges in it (so if A consists just of two adjacent vertices, the A-gap has length $|E(C)| - 1$). We say that A is *normal* in C if every A-gap has even length. An *A-edge* means an edge with both ends in A. If $A, B \subseteq V(C)$, an *(A, B)-gap* is a path P in C between a, b say, where a is the unique vertex of P in A and b is the unique vertex of P in B. (Possibly $a = b$ and P has length 0.)

3.1 *Let C be a cycle, and let $A, B \subseteq V(C)$, such that A, B are both normal. Let P be an odd $A \cap B$-gap. Then P includes an A-gap that contains an odd number of B-edges.*

Proof. Since every B-gap is even, it follows that P contains an odd number of B-edges. But every B-edge of P lies in exactly one A-gap contained in P, so one of the A-gaps in P contains an odd number of B-edges. This proves 3.1. ∎

3.2 *Let C be an odd cycle, and let $A, B \subseteq V(C)$ be normal. Let P be an A-gap. If P contains an odd number of B-edges, then A contains an odd (A, B)-gap and an even one. If P contains an even number of B-edges, then A contains either two odd (A, B)-gaps, or none.*

Proof. Let P have vertices $p_1, \dots, p_n$ say, in order. Since A is normal and C is odd, it follows that $p_1 \neq p_n$ and n is odd. If none of $p_1, \dots, p_n$ belongs to B the claim is true, so we assume that we may choose i, j with $1 \leq i \leq j \leq n$, minimum and maximum respectively such that $p_i \in B$, and therefore P contains exactly two (A, B)-gaps, namely $p_1\text{-}\cdots\text{-}p_i$ and $p_j\text{-}\cdots\text{-}p_n$. The sum of their lengths is $n - j + i - 1$, and since n is odd, it follows that exactly one of the two (A, B)-gaps is odd if and only if $j - i$ is odd. It therefore suffices to show that the number of B-edges in P is odd if and only if $j - i$ is odd. But the B-edges in P are precisely the edges between p_i and p_j that do not lie in B-gaps, and every B-gap is even, so an even number of edges of $p_i\text{-}\cdots\text{-}p_j$ are not B-edges. This proves 3.2. ∎

3.3 *Let C be an odd cycle. Let $A_1, \dots, A_k \subseteq V(G)$ be normal, such that for $1 \leq i < j \leq k$, every (A_i, A_j)-gap is even. Then $A_1 \cap \dots \cap A_k$ is normal.*

Proof. We proceed by induction on k. If $k = 1$ the result is trivial, so we may assume that $k \geq 2$. Define $A_0 = A_1 \cap A_2$. Since there is no odd (A_1, A_2)-gap, 3.1 and 3.2 imply that A_0 is normal.

(1) *For $3 \leq j \leq k$, every (A_0, A_j)-gap is even.*

For suppose not, and choose a path $p_1\text{-}\cdots\text{-}p_n$ ($= P$ say) of C with n even such that p_1 is the unique vertex of P in A_0, and p_n is the unique vertex of P in A_j. Choose h with $1 \leq h \leq n$ maximum such that $p_h \in A_1$, and i with $1 \leq i \leq n$ maximum such that $p_i \in A_2$. Since every (A_1, A_j)-gap is even, it follows that $n - h$ is even, and therefore h is even, since n is even. Since every A_1-gap is even and $p_1, p_h \in A_1$, it follows that there are an odd number of A_1-edges in the path p_1-$\cdots$-p_h. Similarly i is even, and from the symmetry we may assume that $h \le i$. Now none of $p_2, \ldots, p_n \in A_1 \cap A_2$, and in particular no A_1-edge of P is an A_2-edge. Consequently every A_1-edge belongs to a unique A_2-gap contained in P, and therefore some A_2-gap included in C contains an odd number of A_1-edges. By 3.2, there is an odd (A_1, A_2)-gap, a contradiction. This proves (1).

From (1) and the inductive hypothesis applied to $A_0, A_3, \ldots, A_k$, it follows that $A_0 \cap A_3 \cap \cdots \cap A_k$ is normal, and hence $A_1 \cap \cdots \cap A_k$ is normal. This proves 3.3. ∎

We apply this to show the following (this and the previous lemma were proved in joint work with Neil Robertson and Robin Thomas).

3.4 *Let G be a graph, let C be a shortest odd hole in G with no racket, and let X be a stable set of major vertices. Then the set of X-complete vertices in C is normal.*

Proof. Let $X = \{x_1, \ldots, x_k\}$, and for $1 \le i \le k$ let A_i be the set of neighbours of x_i in $V(C)$. For $1 \le i \le k$, since x_i is major and C is a shortest odd hole, it follows that A_i is normal and $|A_i| \ge 3$; and since there is no racket for C, $|A_i| \ge 4$. If for $1 \le i < j \le k$ every (A_i, A_j)-gap is even, then the result follows from 3.3. So we may assume that there is an odd (A_1, A_2)-gap, say P. Let C have vertices $c_1, \ldots, c_{2n+1}$ in order, where P is c_1-$\cdots$-c_r, and r is even, with $2 \le r \le 2n$.

(1) *If Q is an even (A_1, A_2)-gap, then $V(P \cap Q) = \emptyset$ and there is an edge between $V(P), V(Q)$.*

For suppose first that $P \cap Q$ is nonempty, say $c_r \in V(Q)$. Then $P \cup Q$ is an odd A_1-gap or A_2-gap, a contradiction. So $P \cap Q$ is empty. Now assume there are no edges between P and Q. Then their union, together with x_1, x_2, forms an odd hole, of length at most that of C. Since C is a shortest odd hole, this new hole has length equal to that of C, and hence Q is the path c_{r+2}-$\cdots$-c_{2n}. Since $|A_1|, |A_2| \ge 4$ it follows that $c_{r+1}, c_{2n+1} \in A_1 \cap A_2$. But then c_{r+1}-$\cdots$-c_{2n} is either an odd A_1-gap (if $c_{r+2} \in A_2$) or an odd A_2-gap (if $c_{r+2} \in A_1$), in either case a contradiction. This proves (1).

Now there is an even (A_1, A_2)-gap; for if $A_1 \cap A_2$ is normal, then it is nonempty and there is an (A_1, A_2)-gap of length 0, and otherwise there is an even (A_1, A_2)-gap by 3.1 and 3.2. So by (1) we may assume that c_{r+1}-$\cdots$-c_s is an even (A_1, A_2)-gap, for some odd s with $r+1 \le s \le 2n+1$. Since $|A_1| \ge 4$, and only two vertices of $P \cup Q$ belong to A_1, it follows that $s \le 2n-1$. If both A_1, A_2 meet the path c_{s+2}-$\cdots$-c_{2n}, then there is an (A_1, A_2)-gap contained in this path; it is not even, by (1), since there are no edges between it and P, and it is not odd, by (1), since there are no edges between it and Q, a contradiction. So we may assume that none of $c_{s+2}, \ldots, c_{2n}$ belong to A_1, from the symmetry between A_1, A_2. Since $|A_1| \ge 4$, it follows that $c_{s+1}, c_{2n+1} \in A_1$. Choose i, j with $s+1 \le i \le j \le 2n+1$, minimum and maximum respectively such that $c_i, c_j \in A_2$. Then $s+1 \le i < j \le 2n+1$, since $|A_2| \ge 4$. Hence c_{s+1}-$\cdots$-c_i is an (A_1, A_2)-gap, and since there are no edges between it and P, it is odd, that is, i is odd. Similarly c_j-$\cdots$-c_{2n+1} is an (A_1, A_2)-gap, and since there are no edges between it and Q it follows that j is odd. But then c_{s+1}-$\cdots$-c_i and c_j-$\cdots$-c_{2n+1} are odd and even (A_1, A_2)-gaps respectively, violating (1) since $j - i$ is even. This proves 3.4. ∎

4 Anticonnected sets of major vertices

Let C be an odd hole in G, and let X be an anticonnected set of major vertices. We say that X is *well-behaved* if the set of all X-complete vertices in C is normal.

4.1 *Let G be a graph containing no configuration of types $\mathcal{T}_1, \ldots, \mathcal{T}_4$, and such that $\overline{G}$ contains no configuration of type $\mathcal{T}_2$. Let C be a shortest odd hole in G, and assume there is no racket for C. Then every anticonnected set of major vertices is well-behaved.*

Proof. Suppose the result is false, and choose an anticonnected set X of major vertices that is not well-behaved, with X minimal. By 3.4, some two vertices in X are adjacent, and therefore there is an antipath of length ≥ 2 contained in X. Let Q be a maximal antipath in X, between $a, b \in X$ say. Then $X \setminus a, X \setminus b$ are both anticonnected. Let A be the set of all $X \setminus \{a\}$-complete vertices in C, and define B similarly. Then from the minimality of X, A and B are normal, and $A \cap B$ is not, since $A \cap B$ is the set of all X-complete vertices in C.

(1) *There exist vertices in $A \setminus B$ and in $B \setminus A$ that are nonadjacent.*

For assume that every vertex in $A \setminus B$ is adjacent to every vertex in $B \setminus A$. By 3.1 and 3.2, there is an odd $A \cap B$-gap P, containing an odd (A, B)-gap and an even one; and therefore there is an (A, B)-gap in P of length 1, and one of length 0 (and hence $A \cap B \neq \emptyset$). Let C have vertices $c_1, \ldots, c_{2n+1}$ in order. We may assume that $c_1 \in A \setminus B$ and $c_{2n+1} \in B \setminus A$, and c_1, c_{2n+1} both belong to P^*. Choose i, j with $2 \leq i \leq j \leq 2n$, minimum and maximum respectively such that $c_i, c_j \in A \cap B$. Since P is odd, it follows that $j - i$ is even. Suppose that $i > 2$. Since none of $c_2, \ldots, c_{i-1}$ belong to A (since every vertex in $A \setminus B$ is adjacent to c_{2n+1}), it follows that c_1-$\cdots$-c_i is an A-gap, and therefore i is odd. Consequently c_{2n+1}-c_1-$\cdots$-c_i is not a B-gap; but every vertex in $B \setminus A$ is adjacent to c_1, and so $c_2 \in B \setminus A$, and no other internal vertex of c_{2n+1}-c_1-$\cdots$-c_i belongs to B. So c_2-$\cdots$-c_i is an odd path, and since it is not a B-gap it follows that $i = 3$. We have shown then that if $i > 2$ then $i = 3$ and $c_2 \in B \setminus A$. By the same argument, if $j < 2n$ then $j = 2n - 1$ and $c_{2n} \in A \setminus B$. Now not both $c_2 \in B \setminus A$ and $c_{2n} \in A \setminus B$, since every vertex in $A \setminus B$ is adjacent to every vertex in $B \setminus A$; and hence we may assume that $j = 2n$. Since $j - i$ is even it follows that $i \neq 3$, and hence $i = 2$.

Now there are two cases, depending whether some vertex of c_3-$\cdots$-c_{2n-1} belongs to $A \cup B$ or not. Suppose first that, say, $c_h \in A$ for some h with $3 \leq h \leq 2n - 1$. Since c_h is not adjacent to c_{2n+1}, it follows that $c_h \in A \cap B$. But then $c_{2n}, c_{2n+1}, c_1, c_2, c_h$ and the antipath c_1-a-Q-b-c_{2n+1}, form a configuration of type $\mathcal{T}_2$, a contradiction. So none of $c_3, \ldots, c_{2n-1}$ belongs to $A \cup B$. Let Z be the set of all $\{c_2, c_{2n}, c_{2n+1}\}$-complete vertices in G; thus, $X \setminus \{b\} \subseteq Z$. Let Y be the anticomponent of Z that includes $X \setminus \{b\}$. But then $c_{2n}, c_{2n+1}, c_1, c_2$, the path c_2-c_3-$\cdots$-c_{2n}, and Y, form a configuration of type $\mathcal{T}_3$ in G, a contradiction. This proves (1).

(2) *Every vertex of $X \setminus \{a, b\}$ is adjacent to at least one of a, b.*

For suppose that some $x \in X \setminus \{a, b\}$ is nonadjacent to both a, b. Choose $c_i \in A \setminus B$ and $c_j \in B \setminus A$, nonadjacent. Then x-c_i-b-a-c_j-x is a configuration of type $\mathcal{T}_1$, a contradiction. This proves (2).

It follows from (2) that Q has length ≥ 3.

(3) $X \setminus \{a, b\}$ *is anticonnected.*

For assume it has more than one anticomponent; then there is an anticomponent Y of $X \setminus \{a, b\}$ disjoint from Q^*. Since X is anticonnected, there exists $y \in Y$ nonadjacent to one of a, b. Since Y is disjoint from Q^* it follows that y is Q^*-complete, and from the maximality of Q it follows that y is nonadjacent to both a, b, contrary to (2). This proves (3).

(4) *Every odd* (A, B)*-gap has length* 1.

For let C have vertices $c_1, \ldots, c_{2n+1}$ in order, where c_1-$\cdots$-c_r is an odd (A, B)-gap, and assume that $r \geq 3$. We may assume that $c_1 \in A$ and $c_r \in B$. Since $X \setminus \{a, b\}$ is well-behaved, and c_1-$\cdots$-c_r is an odd path and its ends are $X \setminus \{a, b\}$-complete, it follows that some internal vertex of this path is $X \setminus \{a, b\}$-complete, say c_i. But then c_i is nonadjacent to both a, b, since c_1-$\cdots$-c_r is an (A, B)-gap; and therefore c_r, a, b, c_1, c_i, Q is a configuration of type $\mathcal{T}_2$, a contradiction. This proves (4).

By 3.1 there is an A-gap P containing an odd number of B-edges; and by 3.2, there is an odd (A, B)-gap contained in P. By (4) it has length 1. Consequently we may assume that C has vertices $c_1, \ldots, c_{2n+1}$ in order, where $c_1 \in A \setminus B$ and $c_{2n+1} \in B \setminus A$.

(5) *At least one of* $c_3, \ldots, c_{2n-1}$ *belongs to* $(A \setminus B) \cup (B \setminus A)$.

For suppose not. Suppose first that none of $c_3, \ldots, c_{2n-1} \in A \cup B$. Since A, B are normal it follows that $c_2 \in A$ and $c_{2n} \in B$. Let R be the path c_2-c_3-$\cdots$-c_{2n}, and let Y be the anticomponent including Q^* of the set of all $\{c_{2n}, c_1, c_2\}$-complete vertices in G. Since $c_2, c_1, c_{2n+1}, c_{2n}, R, Y$ is not a configuration of type $\mathcal{T}_3$, there is a vertex c_h of R^* that is Y-complete. Since no vertex of R^* is $X \setminus \{a\}$-complete, it follows that $X \setminus \{a\} \not\subseteq Y$, and therefore $c_{2n} \notin A$. Similarly $c_2 \notin B$. But c_h is $X \setminus \{a, b\}$-complete since it is Y-complete, and therefore c_h is nonadjacent to a, b since it does not belong to $A \cup B$. But then $c_{2n+1}, a, b, c_2, c_h, Q$ form a configuration of type $\mathcal{T}_2$, a contradiction.

This proves that there exists h with $3 \leq h \leq 2n-1$ such that $c_h \in A \cup B$, and therefore $c_h \in A \cap B$. If $c_2 \in A \setminus B$, then $c_h, a, c_{2n+1}, c_1, c_2$ and the antipath a-Q-b-c_{2n+1} form a configuration of type $\mathcal{T}_2$, a contradiction. So $c_2 \notin A \setminus B$, and similarly $c_{2n} \notin B \setminus A$. By (1) it follows that $c_2 \in B \setminus A$ and $c_{2n} \in A \setminus B$. From the symmetry, and since $n > 2$ (because G contains no configuration of type $\mathcal{T}_1$), we may assume that $h \geq 4$. But then c_1, c_2, a, c_h, c_{2n} and the antipath c_2-b-Q-a form a configuration of type $\mathcal{T}_2$, a contradiction. This proves (5).

(6) *None of* $c_3, \ldots, c_{2n-1}$ *is* $V(Q)$*-complete.*

For suppose that some c_i is $V(Q)$-complete where $3 \leq i \leq 2n-1$. From (5) there exists h with $3 \leq h \leq 2n-1$ such that $c_h \in (A \setminus B) \cup (B \setminus A)$, and from the symmetry we may assume that $c_h \in A \setminus B$. But then $c_1, c_{2n+1}, a, c_i, c_h$ and the antipath c_{2n+1}-b-Q-a, form a configuration of type $\mathcal{T}_2$, a contradiction. This proves (6).

(7) $X = V(Q)$.

For from (6), no edge of C is $V(Q)$-complete, and therefore $V(Q)$ is not well-behaved. From the minimality of X it follows that $X = V(Q)$. This proves (7).

(8) *Let P be a subpath of c_3-c_4-$\cdots$-c_{2n-1} containing a vertex of $A \cup B$, so that a, b both have neighbours in P. Then P contains an odd (A,B)-gap.*

For choose P minimal such that it contains a vertex of $A \cup B$ and a, b both have neighbours in it. Let its ends be p_1, p_2 say; so one of $p_1, p_2 \in A \cup B$, say $p_1 \in B$, and therefore p_2 is the unique neighbour of b in P. By (6), $p_1 \neq p_2$. Let Y be the anticomponent including Q^* of the set of all $\{c_1, c_{2n+1}, p_1\}$-complete vertices. Since $c_1, c_{2n+1}, a, b, p_1, p_2, P, Y$ do not form a configuration of type $\mathcal{T}_4$, it follows that either some internal vertex of P is Y-complete, or P has length 1 and p_2 is Y-complete. Suppose the first; say c_h is an internal vertex of P, and c_h is Y-complete. It follows that c_h is Q^*-complete, and from the minimality of P, c_h is nonadjacent to both a, b. But $p_1 \in B \setminus A$ by (6), and so p_1, a, b, c_1, c_h, Q is a configuration of type $\mathcal{T}_2$, a contradiction. So there is no such c_h, and hence P has length 1 and p_2 is Y-complete and therefore Q^*-complete. By (6) p_2 is not $V(Q)$-complete, and since p_2 is adjacent to b, it is therefore nonadjacent to a. But then p_2-p_1 is an odd (A,B)-gap, by (7). This proves (8).

(9) *There is an odd (A,B)-gap in c_3-$\cdots$-c_{2n-1}, and $|A \cap B| \leq 1$.*

For by (5), at least one of $c_3, \ldots, c_{2n-1}$ belongs to $A \cup B$. If one of a, b has no neighbour in c_3-$\cdots$-c_{2n-1} (say a), then a is adjacent to both c_2, c_{2n} since it is major, and then $c_1, c_{2n}, c_2, c_{2n+1}, a, c_2$-$c_3$-$\cdots$-$c_{2n}$ is a configuration of type $\mathcal{T}_2$ in $\overline{G}$, a contradiction. So both a, b have neighbours in c_3-$\cdots$-c_{2n-1}, and hence by (8), there is an odd (A,B)-gap contained in c_3-$\cdots$-c_{2n-1}. Since it has length 1 by (4), and by (6) every vertex in $A \cap B$ has a neighbour in each odd (A,B)-gap, the claim follows. This proves (9).

(10) *The number of odd (A,B)-gaps in C is odd.*

For every odd (A,B)-gap in C is contained in a unique A-gap; and every B-edge is in a unique A-gap since there is no $A \cap B$-edge by (9). But by 3.2, an A-gap contains an odd number of B-edges if and only if it includes an odd number of odd (A,B)-gaps. Since C contains an odd number of B-edges (since B is normal), it therefore contains an odd number of odd (A,B)-gaps. This proves (10).

Assign an orientation ("clockwise") to C, where c_2 is the successor of c_1. For every odd (A,B)-gap, let its ends be c, d where d immediately follows c in the clockwise order. Then either $c \in A \setminus B$ and $d \in B \setminus A$, or vice versa. If $c \in A \setminus B$, we say c-d is *white* and otherwise it is *black*. From (9) and (10) there are two (distinct) successive odd (A,B)-gaps that have the same colour. Hence we may assume that for some i with $1 \leq i \leq 2n$, $c_i \in B \setminus A$ and $c_{i+1} \in A \setminus B$, and the path c_1-$\cdots$-c_i contains no odd (A,B)-gap. Since $c_1 \in A \setminus B$ and $c_i \in B \setminus A$, it follows that $i \geq 2$, and similarly $i + 1 \leq 2n$. Moreover, we deduce from (8) applied to the path c_1-$\cdots$-c_i that every odd (A,B)-gap contains one of c_{i+1}, c_{2n+1}. By (10) we may assume that $c_{i+2} \in B \setminus A$, and there are exactly three odd (A,B)-gaps in C.

Suppose that $i + 2 \leq 2n$. Since there are no edges between $\{c_1, \ldots, c_{i-1}\}$ and $\{c_{i+1}, c_{i+2}\}$, it follows from (8) that none of $c_1, \ldots, c_{i-1}$ belong to B. So c_{2n+1}-c_1-$\cdots$-c_i is a B-gap, and therefore i is even. Since c_1-$\cdots$-c_i is not an odd (A,B)-gap, it follows that there exists h with $2 \leq h < i$

such that $c_h \in A$. Choose h maximum. Since c_h-$\cdots$-c_{i+1} is an A-gap, it follows that $i+1-h$ is even, and therefore h is odd. In particular, $h \geq 3$, and so there are no edges between $\{c_h, \ldots, c_i\}$ and $\{c_{2n+1}, c_1\}$; yet c_h-$\cdots$-c_i meets both A and B, and contains no odd (A,B)-gap, contrary to (8). This proves that $i = 2n-1$.

Since A, B are both normal, they both meet c_2-$\cdots$-c_{2n-2}, and yet there are no edges between $\{c_2, \ldots, c_{2n-2}\}$ and $\{c_{2n}, c_{2n+1}\}$ and the path contains no odd (A,B)-gap, contrary to (8). This proves 4.1. ∎

5 The cleaning algorithm

Let a, b, c be vertices of a graph G. We define $D(a,b,c) \subseteq V(G)$ and an integer $r(a,b,c)$ as follows. Let X be the set of all vertices of $G \setminus c$ adjacent to both a, b. If c is X-complete, define $r(a,b,c) = 0$ and $D(a,b,c) = X \cup \{c\}$. If c is not X-complete, define $r(a,b,c)$ to be the cardinality of the largest anticomponent of X that contains a nonneighbour of c, and define $D(a,b,c)$ to be the union of $\{c\}$, all anticomponents of X with cardinality $> r(a,b,c)$, and all anticomponents of X containing a nonneighbour of c.

Now let $v_1, \ldots, v_5 \in V(G)$. We define $N(v_1, \ldots, v_5)$ to be the set of all vertices that either are adjacent to both v_1, v_2, or are adjacent to both v_3, v_4, or are $D(v_3, v_4, v_5)$-complete. The algorithm depends on the following lemma.

5.1 *Let G be a graph containing no configuration of type $\mathcal{T}_1, \ldots, \mathcal{T}_4$, and such that $\overline{G}$ contains no configuration of type $\mathcal{T}_2$. Let C be a shortest odd hole in G and assume that there is no racket for C. Then there exist vertices $v_1, \ldots, v_5$ (not necessarily distinct) such that $N(v_1, \ldots, v_5)$ is a cleaner for C.*

Proof. Let M be the set of all major vertices. Choose vertices $v_3, v_4 \in V(C)$, such that both paths of C joining them have length ≥ 3, and let Y be the set of all vertices adjacent to v_3 and v_4. Let the anticomponents of Y have cardinalities $a_1, \ldots, a_k$ in decreasing order. Choose v_3, v_4 so that a_1 is as large as possible, and subject to that a_2 is as large as possible, and so on.

If $M \subseteq Y$, then the theorem is satisfied by the 5-tuple v_3, v_4, v_3, v_4, v_3, so we assume there exists $v_5 \in M \setminus Y$. Choose v_5 such that $r(v_3, v_4, v_5)$ is as large as possible. Let Z be the set of all $D(v_3, v_4, v_5)$-complete vertices in G. From the choice of v_3, v_4, it follows that no two vertices of $Z \cap V(C)$ have distance > 2 in C. Since C has length > 5 (because G contains no configuration of type $\mathcal{T}_1$), it follows that there is a 2-edge path v_6-v_7-v_8 of C such that $Z \cap V(C) \subseteq \{v_6, v_7, v_8\}$. By 4.1, there is an edge v_1v_2 of C so that v_1, v_2 are both complete to the anticomponent of M that contains v_5. We claim that $v_1, \ldots, v_5$ satisfy the theorem.

For certainly no vertex of C is complete to v_1 and v_2; no vertex of C belongs to $D(v_3, v_4, v_5)$, since all such vertices are adjacent to v_3 and v_4, which have no common neighbour in C; and no vertex of C except possibly v_6, v_7, v_8 is $D(v_3, v_4, v_5)$-complete. So no vertex of C is in $N(v_1, \ldots, v_5)$, except possibly v_6, v_7, v_8. It remains to show that every major vertex belongs to $N(v_1, \ldots, v_5)$. Let y be a major vertex, and suppose that y is not adjacent to both v_3, v_4, and y is not $D(v_3, v_4, v_5)$-complete. Hence $y \notin Y$, and from the choice of v_5, $r(v_3, v_4, y) \leq r(v_3, v_4, v_5)$; and therefore y is complete to every anticomponent of Y that has cardinality $> r(v_3, v_4, v_5)$. But y is not $D(v_3, v_4, v_5)$-complete, since it does not belong to $N(v_1, \ldots, v_5)$, and therefore y is either nonadjacent to v_5 or there is an anticomponent of Y in which both y, v_5 have nonneighbours. In either case it follows that y, v_5 belong to the same anticomponent of M (since $Y \subseteq M$) and therefore y is adjacent to both v_1, v_2, and hence belongs to $N(v_1, \ldots, v_5)$. This proves 5.1. ∎

Now we can prove our main result, 1.1, which we restate.

5.2 *There is an algorithm with the following specifications:*

- **Input** *A graph G.*
- **Output** *Either*
  1. *$O(|V(G)|^5)$ subsets, such that if C is a shortest odd hole in G with no racket, then one of the subsets is a cleaner for C, or*
  2. *it determines that G is not Berge.*
- **Running time** *$(|V(G)|^6)$.*

Proof. First, use the algorithm of 2.8 to test that G and $\overline{G}$ contain no configurations of type $\mathcal{T}_1, \ldots, \mathcal{T}_4$ (actually, in $\overline{G}$ we only need to test for $\mathcal{T}_2$, but it makes no difference to the running time). If we find such a configuration, output that G is not Berge and stop. Otherwise, for each 3-tuple v_3, v_4, v_5, compute $D(v_3, v_4, v_5)$, and list all such sets. Also, for each pair v_1, v_2, compute the set of all vertices adjacent to both v_1, v_2, and list all such sets. Output all subsets that are the union of a set from the first list and one from the second. This proves 5.2. ∎

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